2020 Spring Algebra Prelim

March, 2020

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Please start each solution on a new page and submit your solutions in order.

Notation

$\mathbb{Z}$ the commutative ring of integers.

$\mathbb{Q}$ the field of rational numbers.

$\mathbb{F}_p = \mathbb{Z}/(p)$ the field of p elements where p is a prime.

1. Recall that two $n \times n$ matrices A and B over a field are similar if there is an invertible $n \times n$ matrix Q so that

$$A = QBQ^{-1}.$$

A partition of a positive integer n is a sequence of positive integers $n_1 \geq n_2 \geq \cdots \geq n_k$ so that

$$n_1 + n_2 + \cdots n_k = n.$$

Let $P(n)$ be the number of distinct partitions of n. For example, $P(4) = 5$. Prove that up to similarity of matrices, there are exactly $P(n)$ $n \times n$ matrices A so that $A^n = 0$ (same n).

2. Describe the Galois group and the intermediate fields of the cyclotomic extension $\mathbb{Q}(\zeta_{12})/\mathbb{Q}$.

3.

(a) Prove that $\mathbb{Z}[x]/(2x-1, x^7-1) \cong \mathbb{Z}/(127)$.

(b) Prove that $\mathbb{Z}[i]/(2i-3) \cong \mathbb{F}_{13}$.

4. Let G be the group of matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ with entries in the field $\mathbb{F}_p$.

(a) Prove that G is nonabelian.

(b) Suppose p is old. Prove that $g^p = I_3$ for all $g \in G$.

(c) Suppose that $p = 2$. It is known that there are exactly two nonabelian groups of order 8, up to isomorphism: the dihedral group D_4 and the quaternion group. Assuming this fact without proof, determine which of these groups G is isomorphic to.

5. Let $\mathbb{F}$ be a field and let $\mathbb{F}^\times$ denote the group of nonzero elements of $\mathbb{F}$. Show that every finite subgroup of $\mathbb{F}^\times$ is cyclic.

6.

(a) Let R be a commutative ring with identity which is Noetherian. Let M be a finitely generated R-module, and let $f : M \to M$ be an onto R-homomorphism. Prove that f is an isomorphism.

(b) Give an example showing that without the assumption that M is finitely generated, an onto R-homomorphism $f : M \to M$ is not necessarily an isomorphism.

7. Let G be a finite group, $\mathbb{F}$ an algebraically closed field, and V an irreducible $\mathbb{F}$-linear representation of G.

(a) Show that $\text{Hom}_{\mathbb{F}G}(V, V)$ is a division algebra with $\mathbb{F}$ in its center.

(b) Show that V is finite-dimensional over $\mathbb{F}$, and conclude that $\text{Hom}_{\mathbb{F}G}(V, V)$ is also finite-dimensional.

(c) Show the inclusion $\mathbb{F} \to \text{Hom}_{\mathbb{F}G}(V, V)$ found in (a) is an isomorphism. (For $f \in \text{Hom}_{\mathbb{F}G}(V, V)$, view f as a linear transformation and consider $f - \alpha I$, where α is an eigenvalue of f.)

8. Suppose that R is a ring. Consider an exact sequence of R-modules:

$$(*) \qquad 0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

(Recall that "exactness" means that f is injective, g is surjective, and $\ker(g) = \text{im}(f)$.) We say that the exact sequence (*) splits if there is an R-module homomorphism $h : M_3 \to M_2$ such that $g \circ h : M_3 \to M_3$ is the identity map.

(a) Prove that if the exact sequence (*) splits, then M_2 is isomorphic to the direct sum $M_1 \oplus M_3$.

(b) Suppose R is the polynomial ring $\mathbb{C}[x, y]$ in two indeterminates x and y. Give an example of an exact sequence (*) of R-modules which does not split.

(c) Suppose that R is a commutative integral domain and that every exact sequence (*) of R-modules splits. Prove that R is a field.