Complex Analysis Prelim, Fall 2019

Instructions: Do as many of the eight problems as you can. Four completely correct solutions will be a clear pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. Some problems have more than one part; the parts may not be weighted equally. You may use any standard theorem from your complex analysis text, identifying it either by name or by stating it in full. Be sure to establish that the hypotheses of the theorem are satisfied before you use it. $\mathbb{C}$ denotes the complex numbers, $\mathbb{D}$ the open unit disc $\{z : |z| < 1\}$ and $\mathbb{H}$ the upper half plane $\{z : \Im z > 0\}$.

1. For each case, decide (with proof) whether or not there is a function that is analytic at 0 which takes at the sequence of points $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$, the sequence of values:

   (Case 1) $1, 0, \dfrac{1}{3}, 0, \dfrac{1}{5}, 0, \ldots$

   (Case 2) $1, -\dfrac{1}{4}, \dfrac{1}{9}, -\dfrac{1}{16}, \ldots$

   (Case 3) $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots$

   (Case 4) $1, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{4}}, \ldots$

2. Suppose that f is entire and $|f(z^2)| \leq |f(z)|$ for all $z \in \mathbb{C}$. Prove that f is constant.

3. Show that the infinite product
$$\prod_{n=1}^{\infty} (1 + \frac{1}{n})^z (1 - \frac{z}{n})$$
converges absolutely and locally uniformly to an analytic function on $\mathbb{C}$.

4. Determine all functions $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ that are analytic and injective.

5. Maximize the partial derivative $u_x(0)$ among all harmonic functions $u : \mathbb{D} \to [0, 1]$.

6. If f is analytic in $\mathbb{D}$ and if there are constants $C > 0$ and $0 < \alpha < 1$ such that

$$|f(z) - f(w)| \leq C|z - w|^\alpha$$

for all $z, w \in \mathbb{D}$, show that

$$|f'(z)| \leq C(1 - |z|)^{\alpha - 1}$$

for all $z \in \mathbb{D}$.

7. Consider a sequence of analytic functions $f_n : \mathbb{H} \to \mathbb{H}$. Show that either $|f_n| \to \infty$ uniformly on compact subsets, or there is a subsequence f_{n_k} that converges to an analytic function uniformly on compact subsets.

8. Show that there is no analytic function f in the unit disc $\mathbb{D}$ such that $|f(z_n)| \to \infty$ for all sequences $z_n \in \mathbb{D}$ such that $|z_n| \to 1$.