Instructions: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. Some problems have more than one part; the parts may not be weighted equally.

You may use any standard theorem from your complex analysis text, identifying it either by name or by stating it in full. Be sure to establish that the hypotheses of the theorem are satisfied before you use it.

$\mathbb{C}$ is the complex numbers, $\mathbb{Z}$ is the integers, $\mathbb{D}$ is the open unit disc $\{z : |z| < 1\}$.

1. Evaluate $\displaystyle\int_0^\infty \frac{\cos t}{t^4 + a^4}\, dt$ for $a > 0$.

2. Prove that if $f(z)$ has a pole at z_0 then $e^{f(z)}$ has an essential singularity at z_0.

3. Assume that f is meromorphic on a convex open set U, with poles at $\{z_j\}_{j\in J}$. Show that there exists a meromorphic function g on U satisfying $g'(z) = f(z)$ on $U \backslash \{z_j\}_{j\in J}$ if and only if $\operatorname{Res}(f, z_j) = 0$ for every pole z_j of f.

4. Suppose that f and g are holomorphic on an open set containing $\overline{\mathbb{D}}$, and $f(z) \neq 0$, $g(z) \neq 0$, for all $z \in \partial\mathbb{D}$. Assume that
$$\operatorname{Re}\Big(\frac{f(z)}{g(z)}\Big) \geq 0 \quad \text{for all } z \in \partial\mathbb{D}.$$
Show that f and g have the same number of zeroes in $\mathbb{D}$, counting multiplicity.

5. Prove from first principles (that is, do not assume any properties of the Weierstrass $\wp$-function) that
$$f(z) := \sum_{m,n\in\mathbb{Z}} (z + m + ni)^{-3}$$
converges for $z \in \mathbb{C}\backslash\mathbb{Z}[i]$ to a function that is meromorphic on $\mathbb{C}$. Show that f is $\mathbb{Z}[i]$ periodic, and that all poles of f occur at points in $\mathbb{Z}[i]$ and are all of order 3. Here, $\mathbb{Z}[i]$ is the lattice $\{m + ni : m, n \in \mathbb{Z}\}$.

6. Assume $f(z)$ is analytic on a neighborhood of the annulus $A = \{z : r < |z| < R\}$, where $0 < r < R < \infty$, and assume that $|f(z)| = 1$ for $z \in \partial A$. Show that if f is not constant then f has at least two zeroes in A (counting multiplicity).

7. Let $\{G_n\}_{n=1}^{\infty}$ be an increasing sequence of simply connected open subsets of $\mathbb{C}$, all containing 0, such that $\bigcup_{n=1}^{\infty} G_n$ is not all of $\mathbb{C}$. For each n let f_n be the conformal map of $\mathbb{D}$ onto G_n satisfying $f_n(0) = 0$ and $f_n'(0) > 0$. Prove that:

   (a) $G = \bigcup_{n=1}^{\infty} G_n$ is simply connected.

   (b) The sequence $\{f_n\}$ converges normally in $\mathbb{D}$ (i.e. uniformly on compact subsets) to a conformal map of $\mathbb{D}$ onto G.

8. This problem is about automorphisms of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Recall that an automorphism of $\widehat{\mathbb{C}}$ is a bijective holomorphic map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with holomorphic inverse.

   (a) Prove that for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. the map

   $$z \mapsto \frac{az + b}{cz + d}$$

   is a bijective holomorphism of $\widehat{\mathbb{C}}$. Compute the inverse of this map and show that it is also holomorphic. That is, show that these maps are automorphisms of $\widehat{\mathbb{C}}$.

   (b) Prove that the maps in part (a) give all the automorphisms of $\widehat{\mathbb{C}}$. You may assume that the automorphisms of $\mathbb{C}$ are of the form $z \mapsto az + b$, $a \in \mathbb{C}\backslash\{0\}$.

   (c) Find $a, b, c, d \in \mathbb{C}$ so that the map defined above maps the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ bijectively to

      i. The imaginary axis union ∞:

      $$\{it : t \in \mathbb{R}\} \cup \{\infty\}.$$

      ii. The real axis union ∞:

      $$\{t : t \in \mathbb{R}\} \cup \{\infty\}.$$

      iii. The circle

      $$\{z \in \mathbb{C} : |z - i| = 2\}.$$

   Prove that your answers are correct.

   (d) Determine the collection of $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ so that the map defined in part (a) fixes i.