Topology and Geometry of Manifolds Preliminary Exam
March 2020

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The word "smooth" means C^∞, and all manifolds are assumed to be without boundary unless otherwise specified. Subsets of $\mathbb{R}^n$ are assumed to have the Euclidean topology and standard smooth structure.

Please start each solution on a new page and submit your solutions in order.

1. Let $T^n = \mathbb{S}^1 \times ... \times \mathbb{S}^1$ denote the n-torus, and let M be a connected topological manifold with finite fundamental group. Show that any continuous map from M to T^n is homotopic to a constant map.

2. Let $n \geq 2$. Identify with proof the fundamental group of the following submanifold of $\mathbb{R}^n \times \mathbb{R}^n$:
$$M_n = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \,:\, x \neq y\}.$$

3. Show that
$$M = \{(Y, Z) \in \mathbb{R}^3 \times \mathbb{R}^3 \,:\, Y \times Z = (0, 0, 1)\}$$
is a smoothly embedded submanifold of $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ that is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$, where $\mathbb{S}^1$ is the unit circle. (In $Y \times Z$, the symbol $\times$ indicates the ordinary cross product of vector calculus.)

4. Let $M = \mathbb{R}^n \backslash \{0\}$ and let $||x||$ be the Euclidean norm of $(x^1, x^2, ..., x^n) \in M$. Consider the vector field
$$X = \frac{1}{||x||^n} \sum_{j=1}^{n} x^j \, \frac{\partial}{\partial x^j}$$
on M.

   (a) Find the flow of X. Is X complete?

   (b) Show that the $(n-1)$-form $\omega = i_X(dx^1 \wedge dx^2 \wedge ... \wedge dx^n)$ is closed and use this fact to compute $\mathcal{L}_X(dx^1 \wedge dx^2 \wedge ... \wedge dx^n)$.

5. Let $f : \mathbb{S}^3 \to \mathbb{S}^2$ be a submersion, where $\mathbb{S}^n$ denotes the n-sphere. Prove that f is surjective but has no section.

   (Recall that if $f : M \to N$ is a surjective submersion, a *section* of f is a smooth map $g : N \to M$ such that $f \circ g = Id_N$.)

6. For a smooth manifold M and a nonnegative integer p, let $\Omega^p_c(M)$ denote the vector space of compactly supported smooth p-forms. The pth *compactly supported de Rham cohomology group* $H^p_c(M)$ is the kernel of $d : \Omega^p_c(M) \to \Omega^{p+1}_c(M)$ modulo the image of $d : \Omega^{p-1}_c(M) \to \Omega^p_c(M)$. Prove that for each $n \geq 1$, $H^n_c(\mathbb{R}^n)$ is not the trivial vector space.

7. (a) If ω is a nonvanishing smooth 1-form on a smooth manifold, show that the distribution annihilated by ω is integrable if and only if $\omega \wedge d\omega = 0$.

   (b) If X is a nonvanishing smooth vector field on $\mathbb{R}^3$, prove that the following conditions are equivalent.

      i. Every point in $\mathbb{R}^3$ has a neighborhood U on which there exist smooth functions $f, g : U \to \mathbb{R}$ such that the restriction of X to U is equal to $f \operatorname{grad} g$.

      ii. $\operatorname{curl} X$ is everywhere orthogonal to X.

8. Let M denote the set of unoriented triangles in $\mathbb{R}^3$ with one vertex at the origin. Find a transitive action of a Lie group G on M and use it to identify M with a homogeneous space G/H. Show that this implies that M is naturally a connected smooth manifold, and compute its dimension.