Topology and Geometry of Manifolds Preliminary Exam
September 2019

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The word "smooth" means C^∞, and all manifolds are assumed to be without boundary unless otherwise specified. Subsets of $\mathbb{R}^n$ are assumed to have the Euclidean topology and standard smooth structure.

Please start each solution on a new page and submit your solutions in order.

1. Let $S = [0,1]\times[0,1]$ be the unit square; let $\sim$ be the equivalence relation on S generated by

$$(0,y) \sim (1,y), \qquad 0 \le y \le 1,$$
$$(0,0) \sim (0,1) \sim (1,0) \sim (1,1);$$

and let X be the quotient space $S/{\sim}$. (It looks like a cylinder with two boundary points glued together.) Give a presentation of the fundamental group of X, giving a specific loop representing each generator, and prove your answer correct.

2. Consider the following theorem (which you do not need to prove):

> **Theorem.** *Let M and N be connected smooth manifolds of the same dimension and suppose f is a smooth map from M to N. Also suppose that*
>
> *(a) M is compact,*
>
> *(b) f is surjective, and*
>
> *(c) f has constant rank.*
>
> *Then f is a smooth covering map.*

Exhibit three examples of smooth maps between connected smooth manifolds of the same dimension to show that the conclusion may be false if any one of the hypotheses (a), (b), or (c) fails but the other two hold. Be sure to explain clearly what your examples are, why f is not a smooth covering map, and which hypotheses are satisfied in each case; but you do not have to prove that they satisfy or do not satisfy the required hypotheses.

3. The following multiplication operation makes $\mathbb{R}^3$ into a Lie group (a fact you don't have to prove):

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + e^{z_1} x_2,\ y_1 + e^{-z_1} y_2,\ z_1 + z_2).$$

Find the left invariant vector fields, the one-parameter subgroups, and the Lie algebra.

4. Let $S^2 \subset \mathbb{R}^3$ denote the unit sphere, and let $\iota\colon S^2 \hookrightarrow \mathbb{R}^3$ denote the inclusion map. For each of the following statements, determine whether it is true or false, and prove your claim. (Hint: one is true and the other is false.)

   (a) For every smooth 2-form η on S^2, there is a smooth 2-form $\tilde{\eta}$ on $\mathbb{R}^3$ such that $\eta = \iota^*\tilde{\eta}$.

   (b) For every smooth closed 2-form η on S^2, there is a smooth closed 2-form $\tilde{\eta}$ on $\mathbb{R}^3$ such that $\eta = \iota^*\tilde{\eta}$.

5. For a smooth manifold M and a nonnegative integer p, let $\Omega_c^p(M)$ denote the vector space of compactly supported smooth p-forms. The pth *compactly supported de Rham cohomology group* $H_c^p(M)$ is the kernel of $d : \Omega_c^p(M) \to \Omega_c^{p+1}(M)$ modulo the image of $d : \Omega_c^{p-1}(M) \to \Omega_c^p(M)$. Prove that for each $n \geq 1$, $H_c^n(\mathbb{R}^n)$ is not the trivial vector space.

6. On $\mathbb{R}^4$ with coordinates (w, x, y, z), define 1-forms α and β by

$$\alpha = -dw + 3\,dx + \sin(y^2)\,dy, \qquad \beta = y\,dx - dy + e^x\,dz.$$

   Prove that for each $p \in \mathbb{R}^4$, there exists an embedded 2-dimensional submanifold $S \subset \mathbb{R}^4$ containing p such that $\iota^*\alpha = \iota^*\beta = 0$, where $\iota\colon S \hookrightarrow \mathbb{R}^4$ is the inclusion map.

7. Let $G = \mathbb{R} \times \mathbb{Z}$ act on $\mathbb{R}^2$ by

$$(t, n) \cdot (x, y) = (2^n x,\ y + t + n) \text{ for } (t, n) \in G.$$

   (Here G has the standard product Lie group structure. You don't need to prove this is a group action.)

   Is the orbit space $\mathbb{R}^2/G$ a manifold? Justify your conclusion.

8. Let $M = \mathbb{R}^6 \setminus \{0\}$. Represent a point in M as a pair of vectors in $\mathbb{R}^3$, and define a map $J : M \to \mathbb{R}^3$ by $J(X, Y) = X \times Y$, where "$\times$" indicates the cross product of vector calculus. Consider the subset $C = \{(X, Y) \in M : J(X, Y) = 0\}$.

   (a) Show that the rank of J is constant on C. Explain why this does not allow us to conclude from the Rank Theorem that C is an embedded submanifold of M.

   (b) Show that $D = \{(X, Y) \in C : X \neq 0\}$ is an embedded submanifold of M, diffeomorphic to $S^2 \times \mathbb{R}^+ \times \mathbb{R}$.

   (c) Show that C is an embedded submanifold of M.