# TOPOLOGY AND GEOMETRY OF MANIFOLDS
# PRELIMINARY EXAM

March 27, 2019

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Please start each solution on a new page and submit your solutions in order.

**Note:** *Unless otherwise stated, all manifolds are smooth, connected, and without boundary.*

(1) Represent the Möbius strip as the quotient space of the square $[-1,1]\times[-1,1]$ by identifying $(1,y)$ with $(-1,-y)$ for $-1 \le y \le 1$. Its boundary (as a manifold with boundary) is the image of the set $\{(x,y) \in [-1,1] \times [-1,1] : |y| = 1\}$ under the quotient map. Prove that there does not exist a retraction from the Möbius strip to its boundary.

(2) Let $G$ be a Lie group with identity $e$. Prove that there exists a neighborhood $U$ of $e$ such that each element of $U$ has a unique square root in $U$ (i.e. prove that for each $x \in U$ there exists a unique $v \in U$ with $v^2 = x$.

**Hint:** You will likely need to make use of properties of the exponential map.

(3) Prove that every bounded vector field on $\mathbb{R}^n$ is complete.

(4) Find, with proof, all values of $c \in \mathbb{R}$ so that the locus

$$L_c = \{(x,y) \in \mathbb{R}^2 : y^2 = (x^2-1)(x-c)\}$$

is an embedded submanifold of $\mathbb{R}^2$.

(5) Let $\pi : E \to X$ be a covering map and let $f : Y \to X$ be continuous, where $Y$ is a topological space which is path connected and locally path connected. Set

$$\widetilde{E} = \{(y,e) \in Y \times E : f(y) = \pi(e)\}$$

with the topology induced from the product topology on $Y \times E$. Define $\widetilde{\pi} : \widetilde{E} \to Y$ by

$$\widetilde{\pi}(y,e) = y.$$

Show that if $\widetilde{E}$ is connected, then $\widetilde{\pi}$ is a covering map.

(6) Let $\{(u, v, x, y)\}$ be coordinates on $\mathbb{R}^4$. Show that there are smooth functions, $f_1$ and $f_2$, defined in a neighborhood of 0, with $df_1 \wedge df_2$ nowhere vanishing, and satisfying the system of differential equations

$$\frac{\partial f_j}{\partial x} + y\frac{\partial f_j}{\partial u} + x\frac{\partial f_j}{\partial v} = 0, \qquad \frac{\partial f_j}{\partial y} + x\frac{\partial f_j}{\partial u} + y\frac{\partial f_j}{\partial v} = 0$$

for $j = 1, 2$.

(7) Suppose that $M$ is a $2n$-dimensional manifold that admits a symplectic structure. The latter means that there is a closed differential 2-form $\omega$ such that $\omega^n = \omega \wedge \cdots \wedge \omega$ is nonzero everywhere on $M$.

(a) Show that the even-dimensional deRham cohomology groups $H^{2k}_{dR}(M)$, $k = 1, \ldots, n$, are nontrivial.

(b) Show that the only sphere that admits a symplectic structure is the 2-dimensional sphere $S^2$.

(8) Let $O(n)$ denote the orthogonal group. A *reflection* is a non-identity element $A \in O(n)$ that fixes every point in some linear $(n-1)$-dimensional subspace of $\mathbb{R}^n$. Let $\mathcal{R}_n \subset O(n)$ denote the subset consisting of all reflections. Show that $\mathcal{R}_n$ is a smooth embedded submanifold and is diffeomorphic to the real projective space $\mathbb{RP}^{n-1}$. (Suggestion: It might be useful to consider the action of $O(n)$ on itself by conjugation.)