TOPOLOGY AND GEOMETRY OF MANIFOLDS
PRELIMINARY EXAM

September 13, 2018

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Please start each solution on a new page and submit your solutions in order.

Note: *Unless otherwise stated, all manifolds are smooth, connected, and without boundary.*

(1) Let M^n be a n-dimensional manifold. Let $1 \leq p \leq n$ and let $\omega_1, \dots, \omega_p$ be smooth 1-forms on M which are pointwise linearly independent. Suppose that $\theta_1, \dots, \theta_p$ are smooth 1-forms on M so that

$$\sum_{i=1}^{p} \omega_i \wedge \theta_i = 0.$$

Show that there exist $f_{ij} \in C^\infty(M)$, $1 \leq i, j \leq p$, so that

$$\theta_i = \sum_{i=1}^{p} f_{ij}\omega_j \qquad \text{and} \qquad f_{ij} = f_{ji}.$$

(2) Let M be the set of all straight lines in $\mathbb{R}^2$ (not necessarily passing through the origin).

Show that M can be given the structure of a smooth manifold which is diffeomorphic to $\mathbb{RP}^2$ with one point removed.

(3) Let $\pi : E \to X$ be a covering map and let $f : Y \to X$ be continuous, where Y is a topological space which is path connected and locally path connected. Set

$$\widetilde{E} = \{(y, e) \in Y \times E : f(y) = \pi(e)\}$$

with the topology induced from the product topology on $Y \times E$. Define $\widetilde{\pi} : \widetilde{E} \to Y$ by

$$\widetilde{\pi}(y, e) = y.$$

Show that if $\widetilde{E}$ is connected, then $\widetilde{\pi}$ is a covering map.

(4) Let $T^2 = S^1 \times S^1$ be the 2-torus and let $\mathbb{RP}^2$ be the real projective plane. Prove that any continuous map $f : \mathbb{RP}^2 \longrightarrow T^2$ is homotopic to a constant map.

(5) Find, with proof, all values of $c \in \mathbb{R}$ so that the locus

$$L_c = \{(x, y) \in \mathbb{R}^2 : y^2 = (x^2 - 1)(x - c)\}$$

is an embedded submanifold of $\mathbb{R}^2$.

(6) Consider the 3-form on $\mathbb{R}^4$ given by

$$\alpha = x_1 dx_2 \wedge dx_3 \wedge dx_4 - x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4 - x_4 dx_1 \wedge dx_2 \wedge dx_3.$$

Let S^3 be the unit sphere in $\mathbb{R}^4$ and let $\iota : S^3 \to \mathbb{R}^4$ be the inclusion map.

(a) Evaluate $\int_{S^3} \iota^* \alpha$. Express your answer explicitly as a rational multiple of a power of π.

(b) Let γ be the 3-form on $\mathbb{R}^4 \setminus \{0\}$ defined by

$$\gamma = \frac{\alpha}{(x_1^2 + x_2^2 + x_3^2 + x_4^2)^\lambda}$$

for $\lambda \in \mathbb{R}$. Determine the values of λ for which γ is closed and those for which it is exact.

(7) If G is a Lie group, a subgroup $H \subset G$ is called a discrete subgroup if the induced topology on H is the discrete topology. Suppose G is connected and H is a discrete subgroup which is also a normal subgroup. Show that H is contained in the center of G.

(8) Let V, W be the vector fields on $\mathbb{R}^3$ given by:

$$V = \frac{\partial}{\partial x} + z^2(1 + 4x)\frac{\partial}{\partial y} + 2z\frac{\partial}{\partial z}$$
$$W = 2xz\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

For $p \in \mathbb{R}^3$, set $D_p = \operatorname{span}\{V(p), W(p)\} \subset T_p\mathbb{R}^3$.

(a) Show that D is an involutive rank 2 distribution on $\mathbb{R}^3$.

(b) Find a function u on $\mathbb{R}^3$ so that $\{u = 0\}$ is an integral manifold of D containing the point $(1, 1, 0)$.