Topology and Geometry of Manifolds Preliminary Exam

September 17, 2020

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word "smooth" means C^∞ . Unless otherwise specified, manifolds and associated structures (e.g., maps, vector fields, differential forms) are assumed to be smooth, and manifolds are assumed to be without boundary. Subsets of $\mathbb{R}^n$ are assumed to have the Euclidean topology, and $\mathbb{R}^n$ is assumed to have its standard smooth structure.

(1) Write $\mathbf{x} = (x_1, x_2)$. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x_1, x_2, y) = (x_1|\mathbf{x}|^{-\frac{2}{3}}(y^2 + 1), x_2|\mathbf{x}|^{-\frac{2}{3}}(y^2 + 1), y|\mathbf{x}|^{\frac{4}{3}})$$

when $\mathbf{x} \neq \mathbf{0}$ and $f(0, 0, y) = (0, 0, 0)$. Prove that the graph of f is a smooth embedded submanifold of $\mathbb{R}^6$ of dimension 3. (Hint: Think about why the graph of $x \mapsto x^{\frac{1}{3}}$ is a smooth submanifold of $\mathbb{R}^2$.)

(2) Let X be a connected and locally path-connected space and $p : E \to X$ a two-fold covering map (with E connected).

(a) Prove that $\mathrm{Aut}_p(E)$, the group of deck transformations of p, is isomorphic to $Z/(2)$.

(b) Let ϕ be the nontrivial element of $\mathrm{Aut}_p(E)$. Define

$$E \times_{Z/(2)} \mathbb{R} = E \times \mathbb{R}/\sim,$$

where $(y, -t) \sim (\phi(y), t)$ for all $y \in E$ and $t \in \mathbb{R}$, and let $\pi : E \times_{\mathbb{Z}/(2)} \mathbb{R} \to X$ be given by $\pi(y, t) = p(y)$. Prove that π is naturally a *non-trivial* rank 1 vector bundle over X.

(3) Recall that the torus may be expressed as a smooth embedded submanifold of $\mathbb{R}^3$ consisting of the points

$$T = \left\{ X = (x, y, z) \in \mathbb{R}^3 \;\middle|\; \left(\sqrt{x^2 + y^2} - 2\right)^2 + z^2 = 1 \right\}.$$

Let M be the quotient space obtained by identifying $X \in T$ with its "antipode" $-X$.

(a) Identify — from the classification of compact connected surfaces — the surface to which M is homeomorphic.

(b) Give a finite open cover of M by sets homeomorphic to open subsets of $\mathbb{R}^2$. (You do not need to write down homeomorphisms between your open sets and their homeomorphic images in $\mathbb{R}^2$.) Explain how this open cover may be used, together with a partition of unity argument, to (topologically) embed M in some Euclidean space.

(c) Define a map $F : T \to \mathbb{R}^4$ by
$$F(x, y, z) = (x^2 - y^2, xy, xz, yz).$$
Prove that F descends to a smooth embedding of M in $\mathbb{R}^4$. (You may use without proof the fact that M has a unique smooth structure for which the quotient map $T \to M$ is a submersion.)

(4) Prove that if M is any n-manifold, then the total space TM of the tangent bundle, regarded as a $2n$-manifold, is orientable.

(5) Let X, Y, Z be the vector fields
$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$
$$Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$
$$Z = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$
on $\mathbb{R}^3$, and, for $(a, b, c) \in \mathbb{R}^3 \setminus \{0\}$, consider the vector field $W = aX + bY + cZ$. Let $\Theta : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be the flow of W, and let $\Theta_t : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\Theta_t(p) = \Theta(t, p)$.

Prove that Θ_t is a rotation about the vector $(a, b, c) \in \mathbb{R}^3$; i.e., $\Theta_t \in \mathrm{SO}(3)$ and $\Theta_t(a, b, c) = (a, b, c)$. *In computing Θ_t, you will see that W is complete.*

(6) Let $\omega = f\,dx + g\,dy$ be a closed one-form on $\mathbb{R}^2 \setminus \{0\}$.
(a) If f and g are bounded, prove that ω is exact.
(b) Show by examples that ω may or may not be exact if f and g are not bounded.

(7) Consider the Lie group $G = \mathbb{R}^n \times \mathrm{GL}(n, \mathbb{R})$ with product given by $(v, A) \cdot (w, B) = (v + Aw, AB)$. (This group is isomorphic to the group of affine linear transformations of $\mathbb{R}^n$.) Compute the Lie algebra of G. Your answer should consist of a description of T_eG together with a description of the Lie bracket on this vector space. You may state without proof the Lie algebra of $\mathrm{GL}(n, \mathbb{R})$.

(8) Let G be a Lie group of dimension n and ω a k-form on G. ω is said to be *left invariant* if $(L_g^*)\omega = \omega$ for all $g \in G$ and *right invariant* if $(R_g^*)\omega = \omega$ for all $g \in G$. Here $L_g : G \to G$ and $R_g : G \to G$ denote, respectively, left and right multiplication by g. Prove that, if G is compact and connected and $k = n$, then ω is right invariant whenever it is left invariant. (Hint: For $g \in G$, consider the conjugation map $C_g : G \to G$ given by $C_g(h) = ghg^{-1}$. Prove that $C_g^* : \Omega_e^n(G) \to \Omega_e^n(G)$ is the identity, where e is the identity in G and $\Omega_e^n(G)$ is the fiber over e of the vector bundle of n-forms on G.)