**Algebra Preliminary Exam. September 2006**

**Instructions:** *Do as many problems as you can. Single complete solutions are better than several partial solutions. Correct answers to four problems are a clear pass. Do not reprove major theorems unless asked to do so, but when you use such theorems, say so. In writing down partial solutions try to indicate the gaps as clearly as possible, so that we can see what you do and don't know.*

**1.** Let $\mathbf{Z}$ be the ring of integers and let $\{0\}$ be the trivial $\mathbf{Z}$-module. Let $G$ be a finitely generated $\mathbf{Z}$-module, i.e., a finitely generated Abelian group. Show that if $G \otimes_{\mathbf{Z}} F = \{0\}$ for all fields $F$, then $G = \{0\}$.

**2.** Let $c$ be an automorphism of order 1 or 2 of a field $F$. Suppose that $c$ has the property that for any finite set $\{a_j\}_{j\in J}$ of nonzero elements of $F$, $\Sigma_{j\in J} a_j c(a_j)$ is nonzero. For any $n \times n$-matrix $A = (a_{ij})$ with entries in $F$, let $A^c = (c(a_{ij}))$. The $n \times n$ identity matrix is denoted by $I_n$.

Show that if $A$ has the property that $A(A^c)^T = I_n$, then every eigenvalue $\lambda \in F$ for $A$ satisfies the equation $\lambda c(\lambda) = 1$.

(Remark: This result implies the familiar facts that the eigenvalues of an orthogonal matrix over the field of real numbers are in the set $\{\pm 1\}$ and that the eigenvalues of a unitary matrix over the complex numbers have absolute value 1. One takes $c$ to be the identity map in the first case, complex conjugation in the second.)

**3.** (a). Let $p < q < r$ be prime integers. Show that a group of order $pqr$ cannot be simple.

(b). Consider groups of orders $2^2 \cdot 3 \cdot p$ where $p$ has the values 5,7 and 11. For each of those values of $p$, either display a simple group of order $2^2 \cdot 3 \cdot p$, or show that there cannot be a simple group of that order.

**4.** Let $K/F$ be a finite Galois extension and let $n = [K : F]$. There is a theorem (often referred to as the "normal basis theorem") which states that there exists an irreducible polynomial $f(x) \in F[x]$ whose roots form a basis for $K$ as a vector space over $F$. You may assume that theorem in this problem.

(a) Let $G = Gal(K/F)$. The action of $G$ on $K$ makes $K$ into a finite-dimensional representation space for $G$ over $F$. Prove that $K$ is isomorphic to the regular representation for $G$ over $F$.

(The regular representation is defined by letting $G$ act on the group algebra $F[G]$ by multiplication on the left.)

(b) Suppose that the Galois group $G$ is cyclic and that $F$ contains a primitive $n$-th root of unity. Show that there exists an injective homomorphism $\chi : G \to F^\times$.

(c) Show that $K$ contains a nonzero element $a$ with the following property:

$$g(a) = \chi(g) \cdot a$$

for all $g \in G$.

(d) If $a$ has the property stated in (c), show that $K = F(a)$ and that $a^n \in F^\times$.

**5.** Let $G$ be the group of matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ with entries in the finite field $\mathbf{F}_p$ of $p$ elements, where $p$ is a prime.

(a). Prove that $G$ is nonabelian.

(b). Suppose $p$ is odd. Prove that $g^p = I_3$ for all $g \in G$.

(c). Suppose that $p = 2$. It is known that there are exactly two nonabelian groups of order 8, up to isomorphism: the dihedral group $D_4$ and the quaternionic group. Assuming this fact without proof, determine which of these groups $G$ is isomorphic to.

**6.** Let $R = \mathbf{Z}[x]$, the polynomial ring in a variable $x$ with coefficients in the integers.

(a). Let $M$ be a maximal ideal of $R$. Show that $R/M$ is a finite field.

(b). Suppose that $k$ is any finite field. Prove that there exists at least one and no more than a finite number of maximal ideals $M$ such that $R/M \simeq k$.

(c). Prove that no maximal ideal of $R$ is principal, but that all nonmaximal prime ideals of $R$ are principal.

**7.** There are five nonisomorphic groups of order 8. For each of those groups $G$, find the smallest positive integer $n$ such that there is an injective homomorphism $\phi : G \to S_n$.

**8.** Let $K$ be the field $\mathbf{Q}(z)$ of rational functions in a variable $z$ with coeffients in the rational field $\mathbf{Q}$. Let $n$ be a positive integer. Consider the polynomial $x^n - z \in K[x]$.

(a). Show that the polynomial $x^n - z$ is irreducible over $K$.

(b). Describe the splitting field of $x^n - z$ over $K$.

(c). Determine the Galois group of the splitting field of $x^5 - z$ over the field $K$.