Algebra Preliminary Exam **September 14, 2009**

Instructions: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. (a) Classify groups of order $2009 = 7^2 \times 41$.

   (b) Suppose that F is a field and K/F is a Galois extension of degree 2009. How many intermediate fields are there – that is, how many fields L are there with $F \subset L \subset K$, both inclusions proper? (There may be several cases to consider.)

2. Let K be a field. A discrete valuation on K is a function $\nu : K\backslash\{0\} \to \mathbb{Z}$ such that

   (i) $\nu(ab) = \nu(a) + \nu(b)$

   (ii) ν is surjective

   (iii) $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}\ \forall\ a, b \in K\backslash\{0\}$ with $a + b \neq 0$

   Let $R := \{x \in K\backslash\{0\} : \nu(x) \geq 0\} \cup \{0\}$. Then R is called the valuation ring of ν.

   Prove the following:

   (a) R is a subring of K containing the 1 in K.

   (b) for all $x \in K\backslash\{0\}$, either x or x^{-1} is in R.

   (c) x is a unit of R if and only if $\nu(x) = 0$.

   (d) Let p be a prime number, $K = \mathbb{Q}$ and $\nu_p : \mathbb{Q}\backslash\{0\} \to \mathbb{Z}$ be the function defined by $\nu_p(\frac{a}{b}) = n$ where $\frac{a}{b} = p^n \frac{c}{d}$ and p does not divide c and d. Prove that the corresponding valuation ring R is the ring of all rational numbers whose denominators are relatively prime to p.

3. Let F be a field of characteristic not equal to 2.

   (a) Prove that any extension K of F of degree 2 is of the form $F(\sqrt{D})$ where $D \in F$ is not a square in F and conversely, that each such extension has degree 2 over F.

   (b) Let $D_1, D_2 \in F$ neither of which is a square in F. Prove that $[F(\sqrt{D_1}, \sqrt{D_2}) : F] = 4$ if $D_1 D_2$ is not a square in F and is of degree 2 otherwise.

4. Let F be a field and $p(x) \in F[x]$ an irreducible polynomial.

   (a) Prove that there exists a field extension K of F in which $p(x)$ has a root.

   (b) Determine the dimension of K as a vector space over F and exhibit a vector space basis for K.

   (c) If $\theta \in K$ denotes a root of $p(x)$, express θ^{-1} in terms of the basis found in part (b).

   (d) Suppose $p(x) = x^3 + 9x + 6$. Show $p(x)$ is irreducible over $\mathbb{Q}$. If θ is a root of $p(x)$, compute the inverse of $(1 + \theta)$ in $\mathbb{Q}(\theta)$.

5. Let R be a ring and Q an R-module. According to Baer's criterion, Q is injective if and only if for every ideal I of R, any R-module map $f : I \to Q$ may be extended to an R-module map $g : R \to Q$:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & R \arrow[dl, dotted, "g"] \\ & Q & \end{tikzcd}$$

   (a) Suppose that p is prime and n is a positive integer with p dividing n. Then multiplication makes $\mathbb{Z}/p\mathbb{Z}$ into a module over the ring $\mathbb{Z}/n\mathbb{Z}$. Show that $\mathbb{Z}/p\mathbb{Z}$ is injective as a $\mathbb{Z}/n\mathbb{Z}$-module if and only if p^2 does not divide n.

   (b) Prove that if R is a PID, then an R-module Q is injective if and only if $rQ = Q$ for every nonzero $r \in R$.

6. Fix a ring R, an R-module M, and an R-module homomorphism $f : M \to M$.

   (a) If M satisfies the descending chain condition on submodules, show that if f is injective, then f is surjective. (Hint: note that if f is injective, so are $f \circ f$, $f \circ f \circ f$, etc.)

   (b) Give an example of a ring R, an R-module M, and an injective R-module homomorphism $f : M \to M$ which is not surjective.

   (c) If M satisfies the ascending chain condition on submodules, show that if f is surjective, then f is injective.

   (d) Give an example of a ring R, an R-module M, and a surjective R-module homomorphism $f : M \to M$ which is not injective.

7. Let G be a finite group, k an algebraically closed field, and V an irreducible k-linear representation of G.

(a) Show that $\mathrm{Hom}_{kG}(V, V)$ is a division algebra with k in its center.

(b) Show that V is finite-dimensional over k, and conclude that $\mathrm{Hom}_{kG}(V, V)$ is also finite-dimensional.

(c) Show the inclusion $k \to \mathrm{Hom}_{kG}(V, V)$ found in (a) is an isomorphism. (For $f \in \mathrm{Hom}_{kG}(V, V)$, view f as a linear transformation and consider $f - \alpha I$, where α is an eigenvalue of f.)

8. Recall the following basic definitions and facts about ideals and varieties. Let k be a field and n be a positive integer.

- If $S \subseteq k^n$, the *ideal of* S is $\mathcal{I}(S) := \{f \in k[x_1, \ldots, x_n] : f(s) = 0 \ \forall\ s \in S\}$. $\mathcal{I}(S)$ is a radical ideal in $k[x_1, \ldots, x_n]$.
- If $I \subseteq k[x_1, \ldots, x_n]$ is an ideal, then the *variety of* I in k^n is $\mathcal{V}(I) := \{s \in k^n : f(s) = 0 \ \forall\ f \in I\}$.
- If $S \subseteq k^n$, then $\mathcal{V}(\mathcal{I}(S))$ is the smallest variety containing S and is called the *Zariski closure* of S, denoted as $\overline{S}$.
- Hilbert's Nullstellensatz: If k is algebraically closed and I is an ideal in $k[x_1, \ldots, x_n]$ then $\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}$, where $\sqrt{I}$ is the radical of I.

(a) If I and J are ideals in $k[x_1, \ldots, x_n]$, the *ideal quotient* of I by J is

$$I : J = \{f \in k[x_1, \ldots, x_n] : fg \in I \ \forall\ g \in J\}.$$

You may use without proof the fact that $I : J$ is an ideal in $k[x_1, \ldots, x_n]$ containing I.

Compute $\langle xz, yz\rangle : \langle z\rangle$ in $k[x, y, z]$.

(b) Compute $\mathcal{V}(\langle xz, yz\rangle)$, $\mathcal{V}(\langle z\rangle)$ and $\mathcal{V}(\langle xz, yz\rangle : \langle z\rangle)$.

(c) Let I and J be ideals in $k[x_1, \ldots, x_n]$.

(i) Prove that $\mathcal{V}(I : J) \supseteq \overline{\mathcal{V}(I) \backslash \mathcal{V}(J)}$.

(ii) If k is algebraically closed and $I = \sqrt{I}$ then prove that $\mathcal{V}(I : J) = \overline{\mathcal{V}(I) \backslash \mathcal{V}(J)}$. (Check this statement in the example from parts (a) and (b).)