Algebra Prelim 2013

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Notation: $\mathbf{Q}$ is the field of rational numbers and $\mathbf{C}$ is the field of complex numbers.

Problem 1. Let $\mathbf{Q}^\times$ be the nonzero elements of $\mathbf{Q}$, a group under multiplication.

(a) Prove that the additive group of $\mathbf{Q}$ has no maximal proper subgroups.

(b) Is the same statement true for the multiplicative group $\mathbf{Q}^\times$?

Problem 2. Let V be a finite-dimensional vector space over a field F of characteristic 0. Let $B : V \times V \to F$ be a non-degenerate, skew-symmetric bilinear form. (In particular, we have $B(x, y) = -B(y, x)$ for all $x, y \in V$.) If U is a subset of V, let

$$U^\perp \; = \; \{v \in V \mid B(u, v) = 0 \text{ for all } u \in U \}.$$

(a) Let U be a subspace of V. Prove that $U^\perp$ is a subspace of V and that

$$\dim_F(U) + \dim_F(U^\perp) = \dim_F(V) \; .$$

(b) Prove that there exists a subspace W of V such that $W^\perp = W$.

Problem 3.

(a) Suppose that G is a finitely-generated group. Let n be a positive integer. Prove that G has only finitely many subgroups of index n.

(b) Let p be a prime number. If G is any finitely-generated abelian group, let $t_p(G)$ denote the number of subgroups of G of index p. Determine the possible values of $t_p(G)$ as G varies over all finitely-generated abelian groups.

Problem 4. Suppose that G is a finite group of order 2013. Prove that G has a normal subgroup N of index 3 and that N is a cyclic group. Furthermore, prove that the center of G has order divisible by 11. (You will need the factorization $2013 = 3 \cdot 11 \cdot 61$.)

Problem 5. Let V be a finite dimensional vector space over $\mathbf{C}$. Let $n = dim_{\mathbf{C}}(V)$. Let $T : V \to V$ be a linear map. Suppose that the following statement is true.

For every $c \in \mathbf{C}$, the subspace $\{ v \in V \mid T(v) = cv \}$ of V has dimension 0 or 1.

Prove that there exists a vector $w \in V$ such that $\{w,\ T(w),\ ....,\ T^{n-1}(w)\}$ is a linearly independent set.

Problem 6. This question concerns an extension K of $\mathbf{Q}$ such that $[K : \mathbf{Q}] = 8$. Assume that $K/\mathbf{Q}$ is Galois and let $G = \mathrm{Gal}(K/\mathbf{Q})$. Furthermore, assume that G is nonabelian.

(a) Prove that K has a unique subfield F such that $F/\mathbf{Q}$ is Galois and $[F : \mathbf{Q}] = 4$.

(b) Prove that F has the form $F = \mathbf{Q}(\sqrt{d_1},\ \sqrt{d_2})$, where d_1 and d_2 are nonzero integers.

(c) Suppose that G is the quaternionic group. Prove that d_1 and d_2 are positive integers.

Problem 7. Let $R = \mathbf{C}[x_1, ..., x_n]$ be the polynomial ring over $\mathbf{C}$ in n indeterminates $x_1, ..., x_n$. Let S_n be the n-th symmetric group. If $\sigma \in S_n$, then we can identify σ with the automorphism of R defined as follows: $\sigma(c) = c$ for all $c \in \mathbf{C}$, and $\sigma(x_i) = x_{\sigma(i)}$ for all i, $1 \le i \le n$. Suppose that G is any subgroup of S_n. Let

$$S = R^G = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\} \quad .$$

Prove that S is a finitely-generated $\mathbf{C}$-algebra.

Problem 8. This question concerns the polynomial ring $R = \mathbf{Z}[x, y]$ and the ideal $I = (5, x^2 + 2)$ in R.

(a) Prove that I is a prime ideal of R and that R/I is a PID.

(b) Give an explicit example of a maximal ideal of R which contains I.
(Give a set of generators for such an ideal.)

(c) Show that there are infinitely many distinct maximal ideals in R which contain I.