Complex Analysis Preliminary Exam

Autumn 2006

There are eight problems. Solve as many problems as you can. Four completely correct problems will be a clear pass. Complete problems count more than many problem fragments. Inconclusive argumentation will get little credit so as not to penalize students who are aware of their inability to solve a problem and accordingly write nothing down. Many theorems in complex analysis have several different forms. Make sure that it is clear what the hypotheses of the theorem you are using are and check that you have satisfied all of them.

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In all of these problems $\mathbb{D}$ denotes the (open) unit disc: $\mathbb{D} = \{z : |z| < 1\}$ and $f(\mathbb{D})$ denotes the set $\{w : w = f(z) \text{ for some } z \in \mathbb{D}\}$.

1. Prove that all of the zeros of the polynomial

$$p(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_1 z + c_0$$

lie in the disc centered at 0 with radius

$$R = \sqrt{1 + |c_{n-1}|^2 + \ldots + |c_1|^2 + |c_0|^2}.$$

2. Prove that there exists a sequence of polynomials p_k such that

$$\lim_{k\to\infty} p_k(z) = \begin{cases} 1 & \text{if } \mathrm{Re}(z) > 0 \\ 0 & \text{if } \mathrm{Re}(z) = 0 \\ -1 & \text{if } \mathrm{Re}(z) < 0. \end{cases}$$

3. If f is an analytic map of $\mathbb{D}$ into $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = \frac{1}{2}$, then $f(\mathbb{D})$ contains the disc centered at 0 with radius $7 - 4\sqrt{3}$.
Hint: proof of the Riemann mapping theorem.

4. Let Ω be an open domain in $\mathbb{C}$. We say u is strictly subharmonic on Ω if

$$u(z_0) < \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + \rho e^{i\theta})d\theta$$

for all z_0 and ρ such that $\{z : |z - z_0| \le \rho\} \subset \Omega$. Show that if f is a nonconstant analytic function on Ω then $|f(z)|$ is strictly subharmonic.

5. Let $f(z)$ be an entire function with only finitely many zeroes. Define

$$m(r) = \min_{|z|=r} |f(z)|.$$

Show that if f is not a polynomial then $m(r) \to 0$ as $r \to \infty$.

6. Let $\Omega = \{z : |z| \le 2\}$ and $[0, 1]$ be the line segment from 0 to 1.
(a) Prove that if

(i) $f : \Omega \to \mathbb{C}$ be continuous and

(ii) f is analytic on $\Omega \setminus [0, 1]$

then f is analytic on Ω.

(b) Give an explicit example of a function which is bounded and analytic on $\Omega \setminus [0, 1]$ which can not be extended to an analytic function on Ω. (No credit for using some existence theorem).

7. Let K be a countable closed set of $\mathbb{C}$. Prove that if f is bounded and analytic on $\mathbb{C} \setminus K$ then f is constant.

8. Let $\Omega \subset \mathbb{C}$ be a connected open set with $0 \in \Omega$. Suppose that $U_n(z)$ is a sequence of positive harmonic functions on Ω and $\lim_{n\to\infty} U_n(0) = 0$. Then $U_n(z)$ converges uniformly to 0 on any compact $K \subset \Omega$.