Complex Analysis Preliminary Exam

Autumn 2013

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

In all of these problems, $\mathbb{C}$ will denote the complex plane and $\mathbb{D} = \{z : |z| < 1\}$ will denote the open unit disk. A ***domain*** is an open connected subset of $\mathbb{C}$.

1. Calculate the following integral:
$$\int_0^\infty \frac{\cos x - 1}{x^2}\, dx.$$

2. Suppose $f(z)$ is analytic on the unit disc $\mathbb{D}$ and continuous on the closed unit disc $\overline{\mathbb{D}}$. Assume that $f(z) = 0$ on an arc of the circle $|z| = 1$. Show that $f(z) \equiv 0$.

3. Let $\mathcal{F}$ be the class of holomorphic functions f in the slit half-plane
$$S = \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\} \setminus (0, 1]$$
satisfying $|f(z)| < 1$ for all $z \in S$ and $f(\sqrt{2}) = 0$. Find $f \in \mathcal{F}$ such that $|f(4)| = \sup_{g\in\mathcal{F}} |g(4)|$, and prove your answer correct. (You may write your function as a composition of several other functions, if that is more convenient.)

4. Suppose $\{a_n\}$ is a sequence of distinct complex numbers with no limit point, and $\{b_n\}$ is an arbitrary sequence of complex numbers. Prove that there exists an entire function f with $f(a_n) = b_n$ for $n = 1, 2, 3, \ldots$. (You may use general theorems such as the Mittag–Leffler or Weierstrass theorem without proof, as long as you provide their statements.)

5. Let $f(z)$ be analytic in $\mathbb{D}$ except for finitely many poles. Suppose that $\lim_{z\to e^{i\theta}} |f(z)| = 1$ for every $\theta \in [0, 2\pi)$. Prove that f is a rational function.

6. Suppose f_n is a sequence of holomorphic functions in $\mathbb{D}$ for which $u(z) = \lim \operatorname{Re}(f_n(z))$ exists uniformly on every compact subset of $\mathbb{D}$. Also assume there is a point $z_0 \in \mathbb{D}$ for which $v(z_0) = \lim \operatorname{Im}(f_n(z_0))$ exists. Prove f_n converges uniformly on every compact subset of $\mathbb{D}$.

7. Suppose $\phi\colon [-1, 1] \to \mathbb{C}$ is a continuous function, and define $f\colon \mathbb{C} \setminus [-1, 1] \to \mathbb{C}$ by
$$f(z) = \int_{-1}^{1} \frac{\phi(t)}{t - z}\, dt.$$
Prove that f is holomorphic on $\mathbb{C} \setminus [-1, 1]$, and find an expression for the Laurent coefficients of f about ∞ in terms of ϕ.

8. For any positive integer n, define $f_n\colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by
$$f_n(z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots + \frac{1}{n!z^n}.$$
Let $R > 0$ be given. Show that for sufficiently large n, all of the zeros of f_n lie inside the disk $|z| < R$.