LINEAR ANALYSIS PRELIM EXAM

Autumn 2006

- There are 8 questions. You are guaranteed to pass the exam if you give complete, correct answers to at least **four** of the questions. Partial answers may count but, in general, it is preferable to give complete answers to fewer questions rather than partial answers to more questions.
- If you cannot answer a part of a question, you may assume the result and proceed to a subsequent part.

1. **(a)** What are the possible Jordan forms of a matrix $A \in \mathbf{C}^{4\times 4}$ satisfying $(A-I)^2 = 0$? (Ignore different permutations of the Jordan blocks.)

   (b) Let J_n be an n by n Jordan block with eigenvalue 0. What is the Jordan form of J_n^2?

2. Show that every n by n matrix A is unitarily similar to a matrix with equal diagonal entries; i.e., there exists an n by n matrix U with $U^*U = UU^* = I$ (the identity) such that U^*AU has all of its diagonal entries equal to $\mathrm{tr}(A)/n$ (where $\mathrm{tr}(\cdot)$ denotes the trace). [Hint: One approach is to use induction on the size n of the matrix, together with showing that there is a unit vector $\mathbf{u_1}$ such that $\mathbf{u_1}^* A \mathbf{u_1} = \mathrm{tr}(A)/n$.]

3. Show that there exist infinitely many solutions to the ODE $u'(t) = \sqrt{u(t)}$, $t > 0$, with $u\,(0) = 0$.

4. Consider the initial value ODE problem $u' = f(t,u)$, $0 \le t \le T$, $u(0) = u_0$, where f is C^∞ in t and u. Consider numerical methods of the form

$$u_{i+2} + a_1 u_{i+1} + a_0 u_i = hbf(t_{i+2}, u_{i+2}),$$

   where u_i represents the approximate solution at $t_i = ih$, $h = T/N$.

   (a) Determine the coefficients a_0, a_1, and b that give the highest order local truncation error for the method, and show what that order is.

   (b) Is the resulting method *convergent* (i.e., does the approximate solution converge uniformly to the true solution on the mesh points as $h \to 0$)? Explain why or why not (i.e., either prove your answer directly or quote a theorem and show that all of the hypotheses of the theorem are satisfied).

5. Let X be a Banach space and T a bounded linear operator on X. The *numerical range* of T is the subset of $\mathbf{C}$ defined by

$$W(T) = \{f(Tx) :\ x \in X,\ f \in X^*,\ \|x\| = \|f\| = f(x) = 1\}.$$

   Show that the spectrum of T is a subset of the closure of $W(T)$: $\sigma(T) \subset \overline{W(T)}$.

6. Solve the Cauchy problem

$$\begin{cases} u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & \text{for } r > 0 \\ u(r,\theta) = \sin^3\theta, \quad u_r(r,\theta) = \cos 3\theta & \text{for } r = 1 \end{cases}.$$

7. Find all C^2 solutions to the heat equation

$$u_t - u_{xx} = t - x^2 \quad \text{in} \quad \mathbb{R}^2$$

satisfying

$$\lim_{|x|+|t|\to\infty} \frac{|u(x,t)|}{|x|^5 + |t|^5} = 0.$$

Justify your answer.

8. Find a constant coefficient partial differential operator $L = a\partial_x^2 + b\partial_x\partial_y + c\partial_y^2$ such that $L\chi = \delta$, where χ is the characteristic function of the domain $\{(x,y) \in \mathbf{R}^2 \mid y \geq |x|\}$.