LINEAR ANALYSIS PRELIM EXAM

Autumn 2007

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. The *nuclear norm* of a matrix $A \in \mathbb{R}^{m\times n}$ is defined as the sum of its singular values, i.e.,

$$\|A\|_\nu = \sum_{i=1}^{r} \sigma_i(A),$$

where r is the rank of A. (You do not have to verify that $\|\cdot\|_\nu$ is a norm.) Prove that if $A, B \in \mathbb{R}^{m\times n}$ satisfy $AB^T = 0$ and $A^T B = 0$, then

$$\|A + B\|_\nu = \|A\|_\nu + \|B\|_\nu.$$

(Hint: Use SVD decompositions of A and B to construct a suitable SVD decomposition of $A + B$.)

2. Let T be a bounded linear operator on a Hilbert space H (over $\mathbb{C}$).

   (a) Show that $T = A + iB$, where A and B are self-adjoint operators on H. This is called the *Cartesian decomposition* of T.
   (Hint: Consider linear combinations of T and its Hilbert-space adjoint T^*.)

   (b) Show that the Cartesian decomposition is unique.

   (c) Show that T is normal if and only if A and B commute.

3. Let (X, d) be a metric space and let $g : X \to X$ satisfy the condition

$$d(g(x), g(y)) < d(x, y)$$

for all $x, y \in X$ with $x \neq y$.

   (a) Prove that if (X, d) is compact, then g has a unique fixed point.
   (Hint: Consider $\inf_{x\in X} d(g(x), x)$.)

   (b) Construct an example in which (X, d) is complete but not compact, and $g : X \to X$ satisfies the condition above but has no fixed point.

4. Let C be a subset of a Hilbert space H (over $\mathbb{R}$) endowed with an inner product $\langle \cdot, \cdot \rangle$. Suppose $\{x_n\}$ is a sequence of points in H that is nonexpansive with respect to C in the sense that $\|x_{n+1} - x\| \leq \|x_n - x\|$ for all n and all $x \in C$.

   (a) Prove that $\{x_n\}$ has at most one weak cluster point in C.
   (Hint: If x and x' are two cluster points in C, consider the subsequence limits of $\|x_n - x'\|^2 - \|x_n - x\|^2 \pm \|x' - x\|^2$.)

   (b) Give an example of a set C and a sequence $\{x_n\}$ that is nonexpansive with respect to C, has a weak cluster point in C, but has no strong cluster point. You should verify the example has all the properties. (Hint: Take $C = \{0\}$.)

5. Let $L(t) \geq 0$ be in $L^1[0,1]$, and let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the one-sided generalized Lipshitz condition

$$(\forall t \in [0,1])(\forall y_1 \in \mathbb{R})(\forall y_2 \in \mathbb{R}) \ \left(y_2 > y_1 \implies f(t,y_2) - f(t,y_1) \leq L(t)(y_2 - y_1)\right).$$

Show that there is at most one solution of the initial value problem

$$y'(t) = f(t, y(t)), \qquad y(0) = y_0$$

on the interval $0 \leq t \leq 1$.

6. Let X be a Banach space (over $\mathbb{C}$) and let $T : X \to X$ be a bounded linear operator on X. We say that $\lambda \in \mathbb{C}$ is an *approximate eigenvalue* of T if there exists a sequence $\{x_n\}$ in X with $\|x_n\| = 1$ for all n, and $\|(\lambda I - T)x_n\| \to 0$ as $n \to \infty$.

Suppose that λ is in the continuous spectrum of T. Show that λ is an approximate eigenvalue of T.

7. Consider solving the Cauchy problem

$$\begin{aligned} u_t(x,t) &= u_x(x,t), \qquad (x \in \mathbb{R}, t \geq 0) \\ u(x,0) &= u_0(x), \qquad (x \in \mathbb{R}) \end{aligned}$$

numerically: let $h \equiv \Delta x$, $k \equiv \Delta t$, $\lambda = k/h$, and use the Lax-Wendroff difference scheme

$$u(x,t+k) = \frac{1}{2}(\lambda^2 + \lambda)u(x+h,t) + (1-\lambda^2)u(x,t) + \frac{1}{2}(\lambda^2 - \lambda)u(x-h,t).$$

Assume throughout this problem that the mesh ratio λ is a positive constant as $h, k \to 0$, so for any power p, $O(h^p)$ is equivalent to $O(k^p)$.

(a) Find the order of accuracy of this difference scheme.

(b) Find necessary and sufficient conditions on λ for stability.
(Hint: Express all appearances of the Fourier dual variable ξ in the magnitude squared of the amplification factor in terms of $\sin^2(h\xi/2)$ and simplify.)

8. Let $f \in C\left(\mathbb{R}^n \setminus \{0\}\right)$ satisfy

$$|f(x)| \leq \frac{K}{|x|^m}$$

for some positive constants K and m. Show that there is a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ which agrees with f on $\mathbb{R}^n \setminus \{0\}$.