LINEAR ANALYSIS PRELIM EXAM

Autumn 2012

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. (a) Prove the following variant of the Courant-Fischer min-max principle for singular values: Given an $m \times n$ complex matrix A, $m \geq n$,

$$\sigma_k(A) = \min_{w_1,\dots,w_{k-1}\in\mathbb{C}^n} \max_{\substack{x \in \mathbb{C}^n \\ ||x||_2 = 1 \\ x \perp w_1,\dots,x\perp w_{k-1}}} ||Ax||_2$$

$$= \max_{w_1,\dots,w_{n-k}\in\mathbb{C}^n} \min_{\substack{x \in \mathbb{C}^n \\ ||x||_2 = 1 \\ x \perp w_1,\dots,x\perp w_{n-k}}} ||Ax||_2 \ .$$

(b) Given an $m \times n$ complex matrix A, $m \geq n$, let $1 \leq k \leq n$, and let A_r be a matrix obtained from A by deleting r columns (or rows). Use part a) to prove that for all k and r,

$$\sigma_k(A) \geq \sigma_k(A_r) \geq \sigma_{k+r}(A) \ .$$

Here $\sigma_i(X)$ is the ith singular value of the matrix X, and $||x||_2$ is the two-norm of X, i.e., $||x||_2^2 = x^H x$ for $x \in \mathbb{C}^n$. Finally, X^H represents the Hermitian transpose of X.

2. Given a complex $n \times n$ matrix A, the *Schur decomposition* of A takes the form $A = UTU^H$, where U is unitary and T is upper triangular ($T(i,j) = 0$ for all $n \geq i > j \geq 1$). Here X^H represents the Hermitian transpose of X. The diagonal of T contains all eigenvalues of A, with multiplicities, and it is easy to see that the decomposition is not unique.

   Show that, if A and B are $n \times n$ complex matrices such that $AB = BA$ and A has all distinct eigenvalues, then there exists an $n \times n$ unitary matrix U such that $U^H AU$ and $U^H BU$ are both upper triangular.

3. (a) State and prove the Cauchy-Peano existence theorem for ordinary differential equations.
   (b) Is uniqueness true in the above existence theorem? Justify your answer.

4. Consider the following multistep method

$$y_{n+3} = y_{n+2} + h\left(\beta_2 f(t_{n+2}, y_{n+2}) + \beta_1 f(t_{n+1}, y_{n+1}) + \beta_0 f(t_n, y_n)\right)$$

for numerically approximating the initial values problem $y' = f(t, y)$, $y(0) = y_0$, $y(h) = y_h$, and $y(2h) = y_{2h}$ with stepsize h on the interval $[0, T]$. Here $t_k = kh$ for $0 \leq kh \leq T$ and y_k represents the approximation to $y(t_k)$.

   Assume that y, f are $\mathbb{R}^n$-valued, f is $\mathcal{C}^\infty$ in t and y, and $y_h, y_{2h} \to y_0$ as $h \to 0$.
   Recall that a linear multistep method given by

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\left(\sum_{i=0}^{k} \beta_i f(t_{n+i}, y_{n+i})\right)$$

is zero-stable if the characteristic polynomial $p(r) = \sum_{i=0}^{k} \alpha_i r^i$ has all roots inside or on the unit circle, and those that are on the unit circle are simple.

(a) Find the coefficients $\beta_0, \beta_1, \beta_2$ such that the method is of order 3. (This is known as the "order three Adams-Bashforth method".)

(b) Is the method zero-stable? Is the method convergent?

5. Prove the following (Jacobi) identity:

$$\sum_{k=-\infty}^{\infty} e^{-k^2 t} \frac{e^{ikx}}{\sqrt{2\pi}} = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2t}} e^{-\frac{(x-2\pi k)^2}{4t}} \quad \text{for } t > 0.$$

(Hint: Both sides are periodic functions of x.)

In particular, $\sum_{k=-\infty}^{\infty} e^{-k^2 t} = \sqrt{\frac{\pi}{t}} \left(1 + \sum_{|k|\neq 0} e^{-\frac{\pi^2 k^2}{t}}\right)$.

6. The Radon (or X-ray) transform of a smooth function with compact support on $\mathbb{R}^2$, $f \in C_0^\infty(\mathbb{R}^2)$ is defined as follows: For any line $L_{t,\varphi} = \{x \in \mathbb{R}^2 : \langle x, e^{i\varphi}\rangle = t\}$

$$R(t,\varphi) = \int_L f = \int_{-\infty}^{\infty} f\left(te^{i\varphi} + se^{i\left(\varphi+\frac{\pi}{2}\right)}\right) ds.$$

(a) Suppose two functions $f,\ g \in C_0^\infty(\mathbb{R}^2)$ have the same Radon transforms. Show that $f = g$.

(b) Prove the inversion formula $f(x) = \frac{1}{2i}\int_{S^1} R_t\left(\langle x, e^{i\varphi}\rangle, \varphi\right) d\varphi$.

7. (a) Let

$$l^2 = \left\{ a = (a_1, a_2, \cdots, a_k, \cdots) \mid \sum_{k=1}^{+\infty} a_k^2 < +\infty \right\} \quad \text{and}$$

$$h^2 = \left\{ a = (a_1, a_2, \cdots, a_k, \cdots) \mid \sum_{k=1}^{+\infty} k^2 a_k^2 < +\infty \right\}.$$

Show that $h^2 \subset l^2$ is a compact embedding, namely, that $I : h^2 \to l^2$, $I(a) = a$ is a compact operator.

(b) Is the above operator invertible? Justify your answer.

8. (a) Find all C^2 solutions to the Laplace equation in $\mathbb{R}^2$

$$u_{tt} + u_{xx} = tx$$

satisfying

$$\lim_{|x|+|t|\to\infty} \frac{u(x,t)}{|x|^5 + |t|^5} = 0.$$

(b) Can one have non-polynomial solutions to the wave equation in $\mathbb{R}^2$

$$u_{tt} - u_{xx} = xt$$

satisfying the same growth

$$\lim_{|x|+|t|\to\infty} \frac{u(x,t)}{|x|^5 + |t|^5} = 0?$$