Linear Analysis Prelim Exam

Autumn 2013

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Notation:

- $\mathbb{R}$ (or $\mathbb{C}$) denotes the field of real (or complex) numbers, and $\mathbb{R}^n$ (or $\mathbb{C}^n$) denotes the vector space of n-tuples of real (or complex) numbers.
- $\mathbb{R}^{m\times n}$ (or $\mathbb{C}^{m\times n}$) denotes the space of $m \times n$ matrices with real (or complex) scalars.
- $\mathcal{S}(\mathbb{R}^n)$ denotes the space of Schwartz-class functions on $\mathbb{R}^n$.
- For a vector $v \in \mathbb{R}^n$ or $\mathbb{C}^n$, $|v|$ denotes the Euclidean norm $\left(\sum_{j=1}^n |v_j|^2\right)^{1/2}$.

1. a) Show that the expression

$$v(\phi) = \lim_{N\to\infty} \sum_{\substack{n=-N \\ n\neq 0}}^{N} n^{-1}\phi(n^{-1}), \qquad \phi \in C_c^\infty(\mathbb{R}),$$

defines a distribution on $\mathbb{R}$; that is, $v \in \mathcal{D}'(\mathbb{R})$.

b) Show that there is no constant C such that the following holds for all $\phi \in C_c^\infty(\mathbb{R})$:

$$|v(\phi)| \le C \sup_x |\phi(x)|.$$

c) Find an integrable function $f(x)$ on $\mathbb{R}$ so that $\partial f = v$ as distributions; carefully justify your answer.

2. Consider the differential equation $\partial_t u(t,x) = \partial_x^3 u(t,x)$ on the space $\mathbb{R}^2$ with variables (t,x).

a) Given $f(x) \in \mathcal{S}(\mathbb{R})$, construct a solution $u \in C^\infty(\mathbb{R}^2)$ satisfying

$$\partial_t u(t,x) = \partial_x^3 u(t,x), \qquad u(0,x) = f(x),$$

and such that, for each $t \in \mathbb{R}$, the function $x \to u(t,x)$ belongs to $\mathcal{S}(\mathbb{R})$.

b) If t_n is a sequence converging to 0, show that $u(t_n,x)$ converges to $f(x)$ in the topology on $\mathcal{S}(\mathbb{R})$.

3. Let $D \subset \mathbb{R}^2$ be the disc $|(x, y)| < 1$, and $L^2(D)$ the Hilbert space of square integrable complex-valued functions on D with respect to the usual Lebesgue measure $dx\,dy$.

a) Let $z = x+iy$, and show that the set of functions $\{1, z, z^2, \ldots\}$ form an orthogonal set. Find constants c_n such that $\{c_0, c_1 z, c_2 z^2 \ldots\}$ is an orthonormal set.

b) Let $\mathcal{H} \subset L^2(D)$ denote the closure of the subspace spanned by $\{z^n\}|_{n=0}^{\infty}$. Show that the projection of a given $f \in L^2(D)$ onto $\mathcal{H}$ can be written as a series $\sum_{n=0}^{\infty} a_n z^n$ which converges in $L^2(D)$. Show in addition that this series converges, uniformly on sets $|(x, y)| < r$ for each $r < 1$, to a continuous function on D.
[Remark: Standard results from complex analysis would then show that this continuous function is holomorphic on D. You do *not* have to show this here.]

c) Let T denote the anti-derivative operation on $\mathcal{H}$, defined by

$$T(a_0 + a_1 z + a_2 z^2 + \cdots) = a_0 z + a_1 \frac{z^2}{2} + a_2 \frac{z^3}{3} + \cdots$$

Show that T is a compact map from $\mathcal{H}$ to $\mathcal{H}$, and find its spectrum.

4. Let A be a real $m \times n$ matrix (with $m \geq n$) of rank $r < n$, and let $b \in \mathbb{R}^m$. Define S_b to be the set of all solutions $x \in \mathbb{R}^n$ of the least-squares problem

$$\min_{x \in \mathbb{R}^n} |b - Ax|.$$

Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_n$ be the singular values of A, and let

$$A = U \Sigma V^T$$

be the singular value decompositon (SVD) of A, where $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ are orthogonal matrices, and Σ is the $m \times n$ matrix whose diagonal entries are $\sigma_1, \ldots, \sigma_n$ and whose off-diagonal entries are zero.

a) Let $\Sigma^\dagger$ be the $n \times m$ matrix whose diagonal entries are $1/\sigma_1$, $1/\sigma_2$, $\ldots$, $1/\sigma_r$, 0, $\ldots$, 0 and whose off-diagonal entries are zero. Show that

$$x_* = V \Sigma^\dagger U^T b$$

is in S_b, and in addition, that it is the unique element in S_b of smallest norm.

b) Given (in addition to $b \in \mathbb{R}^m$) an $x_0 \in \mathbb{R}^n$, let x_{0*} be the closest element in S_b to x_0. Express x_{0*} in terms of b, x_0, and the SVD of A.

5. Let $A \in \mathbb{C}^{n \times n}$, and let $\gamma = \max\{Re(\lambda) \,:\, \lambda \text{ is in the spectrum of } A\}$.

Consider the initial-value problem

$$\begin{aligned} \frac{dy}{dt} &= Ay \quad \text{for } t \geq 0, \\ y(0) &= y_0, \end{aligned}$$

where $y : [0, \infty) \to \mathbb{C}^n$. Show that there exists a constant C such that, for any $y_0 \in \mathbb{C}^n$ and for all $t \geq 0$, the solution $y(t)$ of the initial-value problem above satisfies

$$|y(t)| \leq C\left(1 + t^{n-1}\right) e^{\gamma t} |y_0|.$$

6. Let ℓ^2 be the Banach space of all complex sequences $\{x_1, x_2, \ldots\}$ for which $\sum_{k=1}^{\infty} |x_k|^2 < \infty$, with norm

$$\|x\| = \left(\sum_{k=1}^{\infty} |x_k|^2 \right)^{1/2}.$$

a) Construct a bounded linear operator $L : \ell^2 \to \ell^2$ for which

$$\|L\| \equiv \sup_{\{x \in \ell^2 \,:\, \|x\| \neq 0\}} \frac{\|Lx\|}{\|x\|} = 1,$$

but such that for any sequence $x \in \ell^2$ except the zero sequence, $\|Lx\| < \|x\|$.

b) Can an operator L as in part (a) have closed range? Either construct an example or prove that no such operator with closed range exists.

c) Can an operator L as in part (a) be compact? Either construct an example or prove that no such compact operator exists.

7. Let $A \in \mathbb{C}^{n \times n}$ and suppose $\lambda = 1$ is *not* in the spectrum of A. Construct two different matrices $B \in \mathbb{C}^{n \times n}$ for which

$$B^2 - 2B + A = 0.$$

8. Suppose $\psi(h, t, x)$ is real-valued, bounded, continuous in h, t, and x, and uniformly Lipschitz in x on $[0, h_0] \times [a, b] \times \mathbb{R}^n$ for some $h_0 > 0$. Let $f(t, x) = \psi(0, t, x)$, and let $x(t)$ be the solution of the initial-value problem $x'(t) = f(t, x)$, $x(a) = x_a$ on $[a, b]$. For each $h \in (0, \min(h_0, b - a)]$, let $x_0(h) = x_a$, let $t_i(h) = a + ih$ for $0 \le i \le (b - a)/h$, and approximate the solution of the IVP with the one step method

$$x_{i+1}(h) = x_i(h) + h\psi(h, t_i(h), x_i(h)), \quad \text{for} \quad 0 \le i \le \frac{b-a}{h} - 1.$$

Prove that

$$\lim_{h \to 0+} \left[\max_{0 \le i \le (b-a)/h} |x(t_i(h)) - x_i(h)| \right] = 0.$$