LINEAR ANALYSIS PRELIM EXAM

Autumn 2014

- Do as many of the eight problems as you can.
- Four completely correct solutions will be a pass;
- A few complete solutions will count more than many partial solutions. Always carefully justify your answers.
- If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

<u>Notation</u>

- $\mathbb{R}^{n\times m}$ denotes the $n \times m$ real matrices.
- $\mathbb{C}^{n\times m}$ denotes the $n \times m$ complex matrices.
- $\mathbb{S}^n$ denotes the real symmetric matrices
- $H \in \mathbb{S}^n$ is said to be *positive definite* ($H > 0$) if $x^T Hx > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$.
- $H \in \mathbb{S}^n$ is said to be *positive semi–definite* ($H \geq 0$) if $x^T Hx \geq 0$ for all $x \in \mathbb{R}^n$.
- The singular values of a matrix A are the eigenvalues of the matrix $\sqrt{A^*A}$.
- $\mathscr{S}(\mathbb{R}^n)$ denotes the Schwartz space of smooth rapidly decaying functions.
- $\mathscr{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions.
- A function $f \in L^1(\mathbb{R}^n)$ acts as a tempered distribution on $\phi \in \mathscr{S}$ by
$$\langle f, \phi\rangle = \int_{\mathbb{R}^n} f(x)\phi(x)d^n x$$
- $H^2(\mathbb{R}^2)$ is the closure of $C_0^\infty(\mathbb{R}^2)$ in the norm,
$$||u||^2_{H^2(\mathbb{R}^2)} := ||u||^2_{L^2} + ||\Delta u||^2_{L^2}$$
- In $\mathbb{R}^2$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

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1. Let $A \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{m\times m}$, and $V \in \mathbb{R}^{n\times n}$ with W and V symmetric.

(a) Show that V is positive definite on $\ker A$, i.e.,

$$u^T V u > 0 \quad \text{whenever} \quad u \neq 0 \text{ and } u \in \ker A,$$

if and only if there is a $\kappa > 0$ such that the matrix $V + \kappa A^T A$ is positive definite.

(b) Suppose V is positive semidefinite on $\ker A$, i.e.,

$$u^T V u \geq 0 \quad \text{whenever} \quad u \in \ker A.$$

Show that the matrix $M := \begin{bmatrix} V & A^T \\ A & 0 \end{bmatrix}$ is nonsingular if and only if V is positive definite on $\ker A$ and the rank of A is m.

(c) Show that the matrix

$$T := \begin{bmatrix} V & A^T \\ A & W \end{bmatrix}$$

is positive definite if and only if the matrices V and $W - AV^{-1}A^T$ are positive definite.

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2. Let $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$. Show that the three sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ defined below are equal.

$$\begin{aligned} \mathcal{A} &= \{\lambda \in \mathbb{C} \mid \lambda \in \Lambda(X),\ \|A - X\| \leq \epsilon\}, \\ \mathcal{B} &= \left\{\lambda \in \mathbb{C} \;\middle|\; \|(A-\lambda I)^{-1}\| \geq \epsilon^{-1} \text{ or } (A - \lambda I) \text{ is singular.}\right\}, \\ \mathcal{C} &= \{\lambda \in \mathbb{C} \mid \sigma_{\min}(A - \lambda I) \leq \epsilon\}, \end{aligned}$$

where the we have used the operator 2-norm and $\sigma_{\min}(A-\lambda I)$ is the smallest singular value of $(A - \lambda I)$.

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3. Let $m(x) \in C^1([0,1])$ and $\lambda \in \mathbb{C}$. Consider the boundary value problem

$$\begin{aligned} \left(\frac{d}{dx} + m(x) - \lambda\right) u &= f \\ u(0) = u(1) \end{aligned}$$

Let $G(\lambda)$ denote the solution operator as a mapping from $f \in L^2(0,1)$ to $u \in L^2(0,1)$.

(a) Find an explicit formula for $G(\lambda)f$.

(b) Find an explicit formula for the eigenvalues of the boundary value problem (i.e. the values of λ for which $G(\lambda)$ does not exist).

(c) Prove that, if λ is not an eigenvalue, $G(\lambda)$ is a compact operator which maps $L^2(0,1)$ to itself.

4. Suppose $U : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$. If we interpret U as the potential energy of a particle at position x, then $-U'(x)$ (minus the derivative of U) is the force acting on the particle, so (for a particle with mass 1) Newton's law of motion is the second order ODE

$$\frac{d^2x}{dt^2} = -U'(x(t)) \ . \tag{1}$$

For example, for a simple harmonic oscillator (spring), we could have $U(x) = \frac{1}{2}kx^2$ for some $k > 0$.

(a) Rewrite equation (1) as a first-order system.

(b) The kinetic energy of a particle is $\frac{1}{2}(\dot{x})^2$, so the total energy is

$$E(t) = \frac{1}{2}(\dot{x}(t))^2 + U(x(t)) \ .$$

where the dot $\dot{} = \frac{d}{dt}$ means derivative with respect to time. Show that if x solves (1), then $E(t)$ is constant, i.e., energy is conserved.

(c) Suppose that U is bounded from below (that is, there exists $C \in \mathbb{R}$ such that $U(x) \geq C$ for all $x \in \mathbb{R}$). Prove that every solution of (1) exists for all time ($t \to \pm\infty$).

(d) Show that if $U(x) = -x^4$, then the solution of (1) satisfying the initial conditions $x(0) = 0, \ \dot{x}(0) = 1$ blows up in finite time.

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5. Consider the map

$$Mf = \int_0^1 |x - y| f(y) dy$$

mapping $L^2(0,1)$ into continuous (but not necessarily bounded) functions on the real line.

(a) Show that the image of the unit ball in $L^2(0,1)$ are uniformly Lipschitz continuous; i.e.

$$|Mf(a) - Mf(b)| \leq C|a - b|$$

where C depends only on $||f||_{L^2(0,1)}$.

(b) Find the codimension 2 subspace of $L^2(0,1)$ that maps into $L^2(\mathbb{R})$. *Hint: On this subspace, Mf is identically zero outside* $(0,1)$.

(c) Show that M is a compact and injective operator from this subspace into $L^2(\mathbb{R})$.

6. Prove the existence of a solution $u \in H^2(\mathbb{R}^2)$ that satisfies

$$\Delta u - u = F(x, u)$$

under the hypotheses that the $F \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^1)$ satisfies

$$||F(x,0)||_{L^2} \qquad \text{is sufficiently small}$$

$$|F(x,a) - F(x,b)| \quad \leq (|a| + |b|)\, |a - b| \qquad \text{for all real } x, a, b$$

Hint: Use Fourier transform to estimate the H^2 norm of the solution u to the linear PDE $\Delta u - u = f(x)$ in in terms of the L^2 norm of f; then define a mapping from a ball in H^2 to itself, and show its a contraction if $||F(x,0)||_{L^2}$ is small enough. The fact that the supremum ($L^\infty(\mathbb{R}^2)$) norm is bounded by a constant times the $H^2(\mathbb{R}^2)$ norm may be useful. This fact is a form of the Sobolev embedding theorem.

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7. The two definitions below describe how the tempered distributions, r_P and r_D, act on $\phi \in \mathscr{S}$. Prove that the two definitions define the same tempered distribution.

$$\langle r_P, \phi \rangle := \lim_{\epsilon \downarrow 0} \int_{|x| > \epsilon} \frac{1}{x} \phi(x) dx$$

$$\langle r_D, \phi \rangle := \langle \partial_x \log(|x|), \phi \rangle$$

Part of the problem is to prove that each does indeed define a tempered distribution.

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8. (a) Prove that the linear map

$$T\phi = \phi(|x|^2) \qquad x \in \mathbb{R}^2$$

maps $\mathscr{S}(\mathbb{R}^1)$ to $\mathscr{S}(\mathbb{R}^2)$.

(b) By duality, T induces a map

$$T^* : \mathscr{S}'(\mathbb{R}^2) \to \mathscr{S}'(\mathbb{R}^1)$$

Every $f \in \mathscr{S}(\mathbb{R}^2)$ defines a distribution in $\mathscr{S}'(\mathbb{R}^2)$. For such a distribution, $T^* f$ is also a function in $\mathscr{S}(\mathbb{R}^1)$. Find an explicit expression for the function $T^* f$ (i.e. write a formula for its value at every $t \in \mathbb{R}$, not just a formula for its action as a distribution). *Hint: Use polar coordinates*