Topology and Geometry of Manifolds Preliminary Exam

September 15, 2005

Do as many of the eight problems as you can. Four problems done correctly will be a clear pass. Always carefully justify your answers. In partial solutions, it helps to indicate where the gaps are. All manifolds are assumed to be smooth (C^∞) and without boundary, and all structures on them (e.g. differential forms, vector fields) are assumed to be smooth.

1. Suppose X, Y, and Z are path-connected spaces, $p : Y \to Z$ is a covering map, and $f : X \to Z$ is continuous. Let

$$W = \{ (x, y) \in X \times Y : f(x) = p(y) \}.$$

Prove:
a) If $f_* : \pi_1(X) \to \pi_1(Z)$ is surjective, then W is path-connected.
b) If Y is simply-connected, then the converse of part (a) is also true.

2. Let X be the bouquet of two copies of the circle S^1, and let $n > 2$. Give an explicit description of a covering map $p : Y \to X$ such that the image of $p_* : \pi_1(Y, *) \to \pi_1(X, *)$ is a free group on n generators and is a normal subgroup of index $n - 1$. (You may describe Y by drawing a picture, but you must justify why $p : Y \to X$ has the requisite properties.)

3. Using "bump" functions, prove that any compact n-manifold may be smoothly embedded in $\mathbb{R}^N$ for N sufficiently large.

4. Let H be a smooth monoid; that is, H is a manifold and a monoid such that the multiplication map $H \times H \to H$ is smooth. (Recall that a monoid is a set with an associative multiplication and a unit.) Prove that the set of invertible elements in H is open and that the map $g \to g^{-1}$ is smooth on this open subset. (An element is invertible if it has a two-sided inverse.)
Hint: Observe that it suffices to show that the identity has a neighborhood consisting of invertible elements on which the inverse map is smooth.

5. Prove that every bounded vector field on $\mathbb{R}^n$ is complete.

6. Let $w = fdx + gdy$ be a closed one-form on $R^2 - \{0\}$ and suppose that f and g are bounded. Prove that w is exact.

7. Let ω be a nowhere vanishing one-form on the n-manifold M, $n > 1$. Let D be the distribution defined by $D_p = \ker \omega_p$, $p \in M$. Prove that D is integrable if and only if, for every $p \in M$, there is a neighborhood U of p and a smooth nowhere zero real-valued function f on U such that $f\omega$ is exact on U.

8. Show that the Lie algebra of the Lie group $GL(n, \mathbb{R})$ is isomorphic to the vector space $M_n(\mathbb{R})$ of all $n \times n$ real matrices with Lie bracket given by

$$[A, B] = AB - BA,$$

for $A, B \in M_n(\mathbb{R})$.