Topology and Geometry of Manifolds Preliminary Exam
September 15, 2011

Do as many of the eight problems as you can. Four problems done correctly will be a clear pass. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. The word smooth means C^∞ , and all manifolds are assumed to be smooth and without boundary unless otherwise specified.

continuous

1. Let X be a topological space and suppose that $\pi_1(X, x_0)$ is abelian. We say that the two loops $f, g : (S^1, e) \to (X, x_0)$ are freely homotopic if there exists a map $H : S^1 \times [0,1] \to X$ with $H(z,0) = f(z)$ and $H(z,1) = g(z)$ for all $z \in S^1$.*Note that $H(e,t)$ need not be the basepoint x_0 for $0 < t < 1$.*

   Prove that if f and g are freely homotopic, then they are path homotopic.

2. Let $F(x,t)$ be a smooth real valued function defined in a neighborhood of $(0,0)$ in $\mathbb{R}^2$. Suppose that $F(0,0) = F_x(0,0) = F_t(0,0) = 0$, and that the quadratic equation
$$F_{xx}(0,0)w^2 + 2F_{xt}(0,0)w + F_{tt}(0,0) = 0$$
   has distinct real roots, $w_1 \neq w_2$.

   Prove that, inside a possibly smaller neighborhood of $(0,0)$, the level set $\{(x,t) | F(x,t) = 0\}$ is the union of two smooth curves, $(x_1(t), t)$ and $(x_2(t), t)$, which intersect only at $(0,0)$.

   Hint: You may use the fact (which follows from a Taylor expansion with remainder) that
$$F(x,t) = A(x,t)x^2 + 2B(x,t)xt + C(x,t)t^2$$
   with $A(0,0) = \frac{1}{2}F_{xx}(0,0)$, $B(0,0) = \frac{1}{2}F_{xt}(0,0)$, and $C(0,0) = \frac{1}{2}F_{tt}(0,0)$. Look for $x_1(t) = tw_1(t)$ and $x_2(t) = tw_2(t)$.

3. Let X, Y, Z be the vector fields
$$\begin{aligned} X &= z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} \\ Y &= -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z} \\ Z &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \end{aligned}$$
   on $\mathbb{R}^3$, and, for $(a,b,c) \in \mathbb{R}^3$, consider the vector field $W = aX + bY + cZ$. Let $\Theta : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be the flow of W and let $\Theta_t : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $\Theta_t(p) = \Theta(t,p)$.

   Prove that Θ_t is a rotation about the vector $(a,b,c) \in \mathbb{R}^3$; i.e. $\Theta_t \in \mathrm{SO}(3)$ and $\Theta_t(a,b,c) = (a,b,c)$. *In computing Θ_t, you will see that W is complete.*

4. Suppose that M and N are smooth manifolds and that $\pi : M \to N$ is a surjective smooth submersion with connected fibers. We say that a tangent vector $X \in T_pM$ is **vertical** if $d\pi_p(X) = 0$.

   Let $\omega \in \Omega^k(M)$. Show that $\omega = \pi^*\eta$, for some $\eta \in \Omega^k(N)$, if and only if $X \lrcorner\, \omega_p = 0$ and $X \lrcorner\, d\omega_p = 0$ for every $p \in M$ and every vertical vector $X \in T_pM$.

   Hint: Start by proving the result in the special case that $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a projection onto the first n coordinates.

5. Let (M^n, g) denote an oriented Riemannian manifold. For each tangent vector $X_p \in T_p(M)$ we define a cotangent vector $\omega^{X_p} \in \Omega^1_p(M)$ by requiring that

$$\omega^{X_p} = (X_p, \cdot)_g$$

   (a) Show that the formula $(\omega^{X_p}, \omega^{Y_p})^g := (X_p, Y_p)_g$ defines an inner product on the entire vector space $\Omega^1_p(M)$ (i.e. that every $\omega \in \Omega^1_p(M) = \omega^{X_p}$ for a unique X_p, and that $(\omega_1, \omega_2)^g$ is bilinear and positive).

   (b) Let $\{\omega_j\}_{j=1\ldots n}$ be a positively oriented orthonormal frame for $\Omega^1_p(M)$, and define an inner product on $\Omega^k_p(M)$ by declaring that the wedge products $\{\omega_{i_1} \wedge \omega_{i_2} \wedge \ldots \wedge \omega_{i_k}\}_{1 \le i_1 < i_2 < \ldots < i_k \le n}$ form an orthonormal basis for $\Omega^k_p(M)$. The volume form $V^g \in \Omega^n(M)$ is then the unique n-form satisfying, at every p,

$$V^g_p \;:=\; \omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_n$$

   For this problem, you should simply accept this fact that these definitions are independent of the choice of orthonormal frame.

   For a pair of vector fields, X and Y, we define an $(n-2)$-form

$$\eta^{X,Y} := V^g(X, Y, \ldots)$$

   Prove that, for every $\theta \in \Omega^2(M)$,

$$\eta^{X,Y} \wedge \theta = (\omega^X \wedge \omega^Y, \theta)^g \; V^g$$

   (c) Prove that, if X and Y are linearly independent vector fields satisfying $[X, Y] = 0$, then there is a 1-form θ such that $d\eta^{X,Y} = \theta \wedge \eta^{X,Y}$. *This part of the problem does not depend on the previous parts.*

6. Suppose that G and H are Lie groups, that G is connected, and that $\Phi : G \to H$ is a smooth map sending the identity of G to the identity of H, with the property that, for each left invariant vector field X, there exists a (necessarily unique) left invariant vector field Y on H which is Φ-related to X.

   (a) Prove that Φ induces a map $\phi : \mathfrak{G} \to \mathfrak{H}$ of Lie algebras, where $\mathfrak{G}$ is the Lie algebra of G and $\mathfrak{H}$ is the Lie algebra of H.

   (b) Prove that Φ is a Lie group homomorphism. *Hint: Prove that the graph of Φ is a Lie subgroup of $G \times H$. What must the Lie algebra be?*

7. Let F be a smooth mapping from a compact oriented connected d-dimensional manifold M to itself. Suppose that $m \in M$ is a regular ~~point~~ value of F and its inverse image contains exactly N points. Let $\omega \in \Omega^d(M)$ satisfy $\int_M \omega \neq 0$. Show that

connected, oriented

$$\left| \frac{\int_M F^*\omega}{\int_M \omega} \right| \le N$$

   You may assume the fact that a d-form on a d-dimensional manifold which has integral zero is exact. You should prove any facts you use about the degree of a map.

8. Let $p : E \to B$ be a finite sheeted smooth covering map (with E connected), and let G denote the group of deck transformations of p. Suppose that G acts transitively on each fiber. The goal of this problem is to prove that $p^* : H^*_{dR}B \to (H^*_{dR}(E))^G$ is an isomorphism, where $(H^*_{dR}(E))^G$ denotes the G-fixed points of $H^*_{dR}E$.

   (a) If $\omega \in \Omega^k E$, define $\mathrm{tr}(\omega) \in \Omega^k B$ by

$$\mathrm{tr}(\omega)_b(X_1, \ldots, X_k) = \sum_{x \in p^{-1}(b)} \omega_x\left((dp_x)^{-1}X_1, \ldots, (dp_x)^{-1}X_k\right)$$

   Prove that $\mathrm{tr}(\omega)$ is, in fact, smooth, and that $d\mathrm{tr}(\omega) = \mathrm{tr}(d\omega)$.

   (b) Use part a to prove that $p^* : H^*_{dR}B \to (H^*_{dR}(E))^G$ is one-to-one.

   (c) Prove that $p^* : H^*_{dR}B \to (H^*_{dR}(E))^G$ is onto. *Hint: What is $p^*(\mathrm{tr}(\omega))$?*