REAL ANALYSIS PRELIM 2011

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Notation: The set of real numbers is denoted by $\mathbb{R}$. The Lebesgue measure of a set A in $\mathbb{R}$ or $\mathbb{R}^n$ is denoted by $\lambda(A)$, but the Lebesgue integral of a function f is denoted simply by $\int f(x)\,dx$ (where other letters may be substituted for x). If I is an interval in $\mathbb{R}$, $L^p(I)$ denotes the L^p space with respect to Lebesgue measure.

1. Suppose A is a Lebesgue measurable subset of $\mathbb{R}$ with $\lambda(A) > 0$. Show that for all b with $0 < b < \lambda(A)$ there is a compact set $B \subset A$ with $\lambda(B) = b$.

2. Use the fact that $\int_0^\infty e^{-st}t^{n+(1/2)}\,dt = \frac{1}{2}\cdot\frac{3}{2}\cdots(n+\frac{1}{2})\sqrt{\pi}\,s^{-n-(3/2)}$ (which you can assume without proof) to show that
$$\int_0^\infty e^{-st}\sin\sqrt{t}\,dt = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s} \qquad (s>0).$$
Make sure you establish the validity of your calculations.

3. Let $f : [0,1] \to \mathbb{R}$. Suppose that the one-sided derivatives
$$D_-f(x) = \lim_{h<0,\ h\to 0}\frac{f(x+h)-f(x)}{h} \qquad (0 < x \le 1),$$
$$D_+f(x) = \lim_{h>0,\ h\to 0}\frac{f(x+h)-f(x)}{h} \qquad (0 \le x < 1)$$
exist for all x in the indicated ranges and are bounded in absolute value by a constant $K < \infty$. Prove that the (two-sided) derivative $f'(x)$ exists for almost every $x \in (0,1)$.

4. Suppose that f is a nonnegative Lebesgue measurable function on $(0,1]$ such that $\int_0^1 t^3 f(t)^4\,dt < \infty$. Show that
$$\frac{\int_x^1 f(t)\,dt}{|\log x|^{3/4}} \to 0 \text{ as } x \to 0 \quad (x>0).$$
(Hint: First show that $\int_x^1 f(t)\,dt \le C|\log x|^{3/4}$, then refine the argument by writing $\int_x^1 = \int_x^\delta + \int_\delta^1$ for a suitably chosen δ.)

5. Suppose $f \in L^1([0,1])$. Show that if $\int_0^1 f(x)(\sin x)^n\,dx = 0$ for all $n = 1,2,3,\ldots$, then $f = 0$ a.e.

6. Let Λ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Let $\mathcal{G}$ be the family of all sets $A \subset \Lambda$ of the form

$$A = \{ f \in \Lambda : (f(x_1), f(x_2), \ldots, f(x_n)) \in B \}$$

for some finite set $\{x_1, \ldots, x_n\} \subset \mathbb{R}$ ($n = 1, 2, 3, \ldots$) and some open set $B \subset \mathbb{R}^n$ (in the usual Euclidean topology on $\mathbb{R}^n$). Let $\mathcal{T}$ be the topology generated by $\mathcal{G}$, that is, the weakest topology on Λ such that $\mathcal{G} \subset \mathcal{T}$.

a. Show that a sequence $\{f_n\}$ in Λ converges to f with respect to $\mathcal{T}$ if and only if $f_n \to f$ pointwise.
b. Show that the continuous functions are dense in Λ with respect to $\mathcal{T}$.
c. Show that $\mathcal{T}$ is not metrizable — that is, for any metric ρ on Λ, the weakest topology $\mathcal{T}_\rho$ which contains all open balls $\{y \in \Lambda : \rho(x, y) < a\}$, $x \in \Lambda$, $a > 0$, is different from $\mathcal{T}$.

7. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and $T : \mathcal{X} \to \mathcal{Y}$ a one-to-one bounded linear map whose range $T(\mathcal{X})$ is closed in $\mathcal{Y}$. Show that for each bounded linear functional ϕ on $\mathcal{X}$ there is a bounded linear functional ψ on $\mathcal{Y}$ such that $\phi = \psi \circ T$, and there is a constant C (independent of ϕ) such that ψ can be chosen to satisfy $\|\psi\| \le C\|\phi\|$. (Here $\|\phi\| = \sup_{\|x\|=1} |\phi(x)|$, and similarly for $\|\psi\|$.)

8. Let $B(x, r) \subset \mathbb{R}^2$ denote the open disc with center x and radius r and let $S(x, r)$ be the boundary of $B(x, r)$. Let $D = B((0, 0), 1)$ be the unit open disc, and let $\mathcal{H}$ be the family of all bounded Borel measurable functions $f : D \to \mathbb{R}$. Let

$$\mathcal{A} = \left\{ f \in \mathcal{H} : f(x) = \frac{1}{2\pi r} \int_{S(x,r)} f(y)\, d\sigma(y) \text{ for all circles } S(x, r) \subset D \right\},$$

where $d\sigma(y)$ denotes the arc length measure on $S(x, r)$, and

$$\mathcal{B} = \left\{ f \in \mathcal{H} : f(x) = \frac{1}{\pi r^2} \int_{B(x,r)} f(y)\, dy \text{ for all discs } B(x, r) \subset D \right\},$$

where dy denotes 2-dimensional Lebesgue measure. Prove that $\mathcal{A} = \mathcal{B}$. (It may be useful to show that all functions in $\mathcal{B}$ are continuous.)