REAL ANALYSIS PRELIMINARY EXAM

AUTUMN 2014

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. In case of a partial proof, point out the gap and, if possible, indicate what would be required to fill it in. You may use any standard theorem from your real analysis course, either identifying it by name or stating it in full.

Problem 1. Let K be the family of all non-empty compact subsets of $\mathbb{R}$. For $A, B \in K$, let

$$d(A,B) = \max\left(\sup_{y\in A} \inf_{x\in B} |x-y|,\ \sup_{y\in B} \inf_{x\in A} |x-y| \right).$$

Assume without proof that d is a metric on K. Prove that K with the metric d is complete.

Problem 2. Suppose that $f : (0,1) \to \mathbb{R}$ is a continuous function with the property that for every $x \in \mathbb{R}$, the set $f^{-1}(x)$ has at most $(f(x))^2$ elements. Prove that for almost every (with respect to Lebesgue measure) $x \in [0,1]$ the derivative $f'(x)$ exists and $|f'(x)| < \infty$.

Problem 3. Suppose that $f_n : \mathbb{R} \to [0,1]$ for $n = 1, 2, \ldots$, and that each one of these functions is non-decreasing, that is, $f_n(x) \le f_n(y)$ if $x \le y$, for all n, x and y. Prove that there exist a function $g : \mathbb{R} \to [0,1]$, a finite or countably infinite set $A \subset \mathbb{R}$, and a subsequence f_{n_k} such that $\lim_{k\to\infty} f_{n_k}(x) = g(x)$ for all $x \in \mathbb{R} \setminus A$.

Problem 4. Let $L^1(\mathbb{R}^d)$ be the set of Lebesgue integrable functions in $\mathbb{R}^d$, let m be the Lebesgue measure in $\mathbb{R}^d$ and $\|f\|_1 = \int_{\mathbb{R}^d} |f|\,dm$. Let $\{f_n\}_{n\ge1}$ be a sequence in $L^1(\mathbb{R}^d)$. Suppose that there exists $f \in L^1(\mathbb{R}^d)$ such that $f_n \to f$ m-a.e. and $\|f_n\|_1 \to \|f\|_1$.

(1) Show that for all Lebesgue sets $E \subset \mathbb{R}^d$

$$\int_E |f_n|\,dm \to \int_E |f|\,dm \text{ as } n \to \infty.$$

(2) Prove that

$$\|f_n - f\|_1 \to 0 \text{ as } n \to \infty.$$

(3) Show that the sequence $\{f_n\}_{n\ge1}$ converges weakly to f in $L^1(\mathbb{R}^d)$.

Problem 5. Suppose that $K \in L^p([0,1] \times [0,1])$ for some $1 < p < \infty$. Let q be the conjugate exponent of p, that is $1/p + 1/q = 1$.

(1) For $f \in L^q([0,1])$ let
$$(Af)(x) = \int_0^1 K(x,y) f(y)\, dy.$$
Show that $(Af)(x)$ exists for almost every (with respect to the 1-dimensional Lebesgue measure) $x \in [0,1]$. Prove that A is a bounded linear map from $L^q([0,1])$ into $L^p([0,1])$, i.e. $A \in \mathcal{L}(L^q([0,1]), L^p([0,1]))$.

(2) Suppose that for every $f \in L^q([0,1])$, $(Af)(x) = 0$ for almost every (with respect to the 1-dimensional Lebesgue measure) $x \in [0,1]$. Show that $K = 0$ for almost every (with respect to the 2-dimensional Lebesgue measure) $(x,y) \in [0,1] \times [0,1]$.

Problem 6. Let $\mathcal{B}$ be a real Banach space. Denote by $\|\cdot\|$ the norm on $\mathcal{B}$. Suppose that $\mathcal{B}$ satisfies the best approximation property, that is:

Given a closed subspace $\mathcal{M}$ of $\mathcal{B}$ and given $x \in \mathcal{B}$ there exists $y_x \in \mathcal{M}$ such that $\|x - y_x\| = \inf_{y \in \mathcal{M}} \|y - x\| = \operatorname{dist}(x, \mathcal{M})$.

Prove that for every bounded linear function $f \in \mathcal{B}^*$ there exists $z \in \mathcal{B}$ such that $f(z) = \|f\| \|z\|$.

Problem 7. Let $\mathcal{X}$ be a Banach space; let $\mathcal{Y}$ and $\mathcal{Z}$ be normed vector spaces. Let $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$ and $\|\cdot\|_{\mathcal{Z}}$ denote the respective norms. Note that $\mathcal{Y}$ and $\mathcal{Z}$ are not assumed to be Banach spaces. Let $B : \mathcal{X} \times \mathcal{Y} \to Z$ be a bilinear map such that
$$B(x, \cdot) \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}) \text{ for all } x \in \mathcal{X}$$
and
$$B(\cdot, y) \in \mathcal{L}(\mathcal{X}, \mathcal{Z}) \text{ for all } y \in \mathcal{Y}.$$
Here $\mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $\mathcal{L}(\mathcal{X}, \mathcal{Z})$ denote the families of all bounded linear maps from $\mathcal{Y}$ to $\mathcal{Z}$ and from $\mathcal{X}$ to $\mathcal{Z}$ respectively.

(1) Show that there exists a constant $C > 0$ such that
$$\|B(x,y)\|_{\mathcal{Z}} \le C \|x\|_{\mathcal{X}} \|y\|_{\mathcal{Y}} \text{ for all } x \in \mathcal{X},\ y \in \mathcal{Y}.$$

(2) Prove that the displayed statement implies that B is continuous.

PROBLEM 8 IS STATED ON THE NEXT PAGE

Problem 8. (1) Let $\mathbf{1}_D(x)$ denote the characteristic function of a set D, that is, $\mathbf{1}_D(x) = 1$ if $x \in D$ and $\mathbf{1}_D(x) = 0$ if $x \notin D$. Suppose that the Lebesgue measure of a set $D \subset [0,1]$ is greater than or equal to a. Prove that

$$\int_0^1 \frac{2\sqrt{3}}{3}\, x\, \mathbf{1}_D(x)\, dx \geq \frac{\sqrt{3}}{3} a^2.$$

(2) Let T be an equilateral triangle with the unit height (see Figure 1). Let A, B and C denote the three edges and let H_A, H_B and H_C denote the three heights. Let F_A be a subset of the family of all straight lines parallel to A. Similarly, let F_B be a subset of the family of all straight lines parallel to B and let F_C be a subset of the family of all straight lines parallel to C. Let $K_A \subset H_A$ be the image of $\bigcup_{L \in F_A} L$ via orthogonal projection onto H_A and let K_B and K_C be defined in an analogous way. Let m_A denote the Lebesgue measure (length measure) on H_A and and let m_B and m_C have analogous meaning.

Prove that if $m_A(K_A) > \sqrt{2/3}$, $m_B(K_B) > \sqrt{2/3}$ and $m_C(K_C) > \sqrt{2/3}$ then there exist lines $L_A \in F_A$, $L_B \in F_B$ and $L_C \in F_C$ such that $L_A \cap L_B \cap L_C \neq \emptyset$.

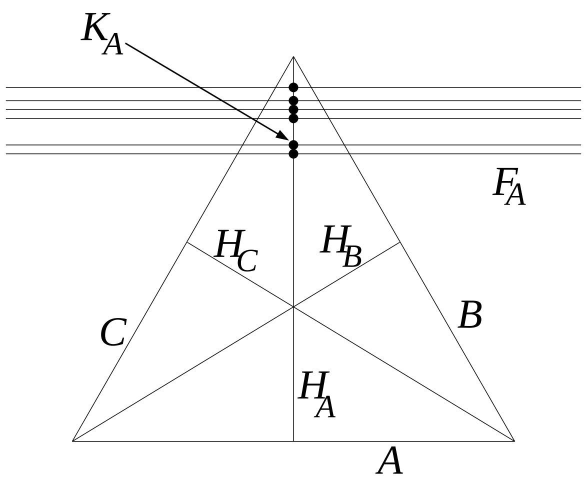

FIGURE 1. The figure is schematic. The sets F_A and K_A are not finite under the assumptions stated in the problem.