REAL ANALYSIS PRELIM 2019

Abstract. Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. You may use any standard theorem from your real analysis course, either identifying it by name or stating it in full.

Note: *In all problems, functions in $\mathbf{L}^p$ spaces are real-valued.*

Problem 1. Suppose that every f in a closed linear subspace M of $\mathbf{L}^\infty(-1,1)$ is continuous in some neighborhood of 0. Prove that there exists a fixed neighborhood of 0 such that every $f \in M$ is continuous in that neighborhood.

Problem 2. Let H_1 and H_2 be Hilbert spaces with scalars in $\mathbb{R}$. Let $T : H_1 \to H_2$ be a linear isometry from H_1 into (but not necessarily onto) H_2, i.e., T is linear and $\|Tx\| = \|x\|$ for all $x \in H_1$. Prove that $\langle Tx, Ty\rangle = \langle x, y\rangle$ for all $x, y \in H_1$.

Problem 3. Let $\mathbf{C}[0,1]$ be the Banach space of all real-valued continuous functions on $[0,1]$, equipped with the uniform norm $\|f\|_u = \sup_{x\in[0,1]} |f(x)|$. Recall that the dual space of $\mathbf{C}[0,1]$ is the space M of all finite signed Borel measures on $[0,1]$. Let X be the subset of M consisting of all probability measures on $[0,1]$ (i.e., positive Borel measures μ with $\mu[0,1] = 1$), equipped with the weak* topology (i.e., the subspace topology on X inherited from the weak* topology on M).

For $\mu \in X$, let the support of μ, denoted by $\mathrm{supp}(\mu)$, be the set of all $x \in [0,1]$ for which every open neighborhood of x has positive μ measure. Show that for any $\mu \in X$ and any open subset U of $[0,1]$ such that $\mathrm{supp}(\mu) \cap U \neq \emptyset$, the subset of X

$$\{\nu \in X : \ \mathrm{supp}(\nu) \cap U \neq \emptyset\}$$

is an open neighborhood of μ in X in the weak* topology.
(This property is called *lower hemi-continuity* of the correspondence $\mu \mapsto \mathrm{supp}(\mu)$.)

Problem 4. Let A and B be bounded Lebesgue measurable subsets of $\mathbb{R}$, both with positive measure. Let χ_A and χ_B be the characteristic functions of A and B.

(a) Show that the convolution $\chi_A \star \chi_B$ is a continuous function and $\int_{\mathbb{R}}(\chi_A \star \chi_B) > 0$.

(b) Show that $A+B := \{x + y \,:\, x \in A,\, y \in B\}$ contains a nonempty open interval.

Problem 5. Let $1 \le p \le \infty$. Let $f_n \in \mathbf{L}^p([0,1])$ be such that $\|f_n\|_p \le n^{-\alpha}$ for some $\alpha > 1$. Let $g_N = \sum_{n=1}^N f_n$. Show that g_N converges in measure as $N \to \infty$.

Problem 6. Let p and q satisfy $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.
Suppose that $k(s,\cdot)$ is a continuous map from a compact set $\mathbf{\Omega} \subset \mathbb{R}^n$ into $\mathbf{L}^q(\mathbb{R}^n)$ (*i.e. for every $s \in \mathbf{\Omega}$, $k(s,t)$, as a function of $t \in \mathbb{R}^n$, is in $\mathbf{L}^q(\mathbb{R}^n)$, and, as s_1 approaches s_2, $k(s_1,t)$ converges in $\mathbf{L}^q(\mathbb{R}^n)$ to $k(s_2,t)$*). Let $\mathbf{K}$ denote the mapping

$$f(t) \overset{\mathbf{K}}{\mapsto} \int_{\mathbb{R}^n} k(s,t) f(t)\,dt,$$

which maps functions f of $t \in \mathbb{R}^n$ into functions of $s \in \mathbf{\Omega}$.

(a) Show that $\mathbf{K}$ maps $\mathbf{L}^p(\mathbb{R}^n)$ into $\mathbf{C}(\mathbf{\Omega})$, the Banach space of continuous real-valued functions on $\mathbf{\Omega}$, equipped with the uniform norm.

(b) Show that the image under $\mathbf{K}$ of the closed unit ball in $\mathbf{L}^p(\mathbb{R}^n)$ has compact closure in $\mathbf{C}(\mathbf{\Omega})$.

Problem 7. Let $\mathbf{C}(\mathbb{R}^n)$ be the space of real-valued continuous functions on $\mathbb{R}^n$. Let K_j for $j \geq 1$ be an increasing sequence of compact subsets of $\mathbb{R}^n$ that exhaust $\mathbb{R}^n$, i.e., each K_j is contained in the interior of K_{j+1} and $\bigcup_{j\geq 1} K_j = \mathbb{R}^n$.

For f and g in $\mathbf{C}(\mathbb{R}^n)$, let

$$\|f-g\|_j = \sup_{x\in K_j} |f(x)-g(x)| \quad \text{for} \quad j \geq 1,$$

and define

$$d(f,g) := \sum_{j=1}^{\infty} \frac{1}{2^j}\left(\frac{\|f-g\|_j}{1+\|f-g\|_j}\right).$$

Prove that d is a metric on $\mathbf{C}(\mathbb{R}^n)$, and that convergence of a sequence $\{f_m\}_{m\geq 1}$ in $\mathbf{C}(\mathbb{R}^n)$ with the metric d is equivalent to uniform convergence of f_m on all compact sets of $\mathbb{R}^n$.

Problem 8. Let $1 \leq p < \infty$. For every $f \in \mathbf{L}^p([0,1])$, and every integer $n \geq 1$, define the piecewise constant function f_n (which depends on both n and f) by

$$f_n(x) = \begin{cases} n\displaystyle\int_{k/n}^{(k+1)/n} f(x)dx \quad \text{for} \quad \dfrac{k}{n} \leq x < \dfrac{k+1}{n} \quad \text{when} \quad 0 \leq k \leq n-2, \\ n\displaystyle\int_{k/n}^{(k+1)/n} f(x)dx \quad \text{for} \quad \dfrac{k}{n} \leq x \leq \dfrac{k+1}{n} \quad \text{when} \quad k = n-1. \end{cases}$$

(a) Show that for every $f \in \mathbf{L}^p([0,1])$, $\|f_n\|_p \leq \|f\|_p$.

(b) Show that for every continuous f, f_n converges uniformly to f.

(c) Use parts (a) and (b) and the density of continuous functions in $\mathbf{L}^p([0,1])$ to show that, for any $f \in \mathbf{L}^p([0,1])$, the sequence $\{f_n\}_{n\geq 1}$ converges to f in $\mathbf{L}^p([0,1])$.