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A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order;

By N. ARONSZAJN (1).

1. A measurable function u defined in a domain D of the euclidean n-space $\mathcal{E}^n$ is said to have a zero of order $\alpha > 0$ at $x_0 \in D$ in the p-mean ($1 \leq p \leq \infty$) if

$$\int_{|x-x_0|<r} |u|^p \, dx = O(r^{p\alpha+n}).$$

If $p < q$, a zero in q-mean is *a fortiori* a zero in p-mean. The usual definition of a zero of order α [i. e. that except on a set of measure 0 $|u(x)| = O(|x - x_0|^\alpha)$ for $x \to x_0$] corresponds to a zero in p-mean for $p = \infty$. The zero is said to be of infinite order if it is of order α for all $\alpha > 0$.

Consider solutions u of an inequality of the following kind

$$(1.1) \qquad |A u(x)|^2 \leq M \left\{ \sum_1^n \left| \frac{\partial u(x)}{\partial x^i} \right|^2 + |u(x)|^2 \right\}.$$

Here A denotes a linear elliptic differential operator of second order, in general with variable coefficients, and M a positive constant.

Our purpose in this Note is to prove the following theorem for solutions of (1.1).

Theorem. — *If u is a solution of* (1.1) *in a domain* D *and if at some*

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point x_0 in D u has a zero of infinite order in the 1*-mean, then u vanishes identically.*

We state below the restrictions on A and u under which we prove the theorem.

1° The principal part of A, $a^{ij}\frac{\partial^2}{\partial x^i \partial x^j}$ should have coefficients a^{ij} of class $C^{(2,1)}$ (i. e. of class C^2 and with second derivatives lipschitzian), all other coefficients of A being uniformly bounded. We can and will suppose (because of ellipticity) that $\{a^{ij}\}$ is a positive definite matrix everywhere in D.

2° u is in the neighborhood of every point of D, the restriction of a potential of order 2 of an L^2-function, the potential vanishing outside of a compact in $\mathcal{E}^n$. Essentially, this is equivalent to the fact that locally u has strong L^2-derivatives of first and second order (²). With this general assumption on u, (1.1) will be assumed to be true only almost everywhere in D.

The validity of the theorem for domains of $\mathcal{E}^n$ implies, obviously, its validity on a differentiable manifold. a^{ij} is then a contravariant tensor and in view of 1° we shall have to assume that the manifold is at least of class $C^{(3,1)}$.

For the case of two dimensions, the theorem is essentialy due to T. Carleman [1]. It was proved recently for n-dimensions by C. Muller [2] and E. Heinz [3], the operator A being the ordinary laplacian, and the function n being subject to somehow stronger restrictions. C. Muller considered, essentially, the inequalities of type (1.1) with right hand side reduced to $M|u(x)|^2$, whereas Heinz considered them in the form given above. Both authors made extensive use of spherical harmonics in their proofs (³).

Heinz reduces the proof to the following inequality

$$(1.2)\qquad cr^2\int_{|x|<r}|x|^{-2\alpha}|\Delta u|^2\,dx \geq \int_{|x|<r}|x|^{-2\alpha}\left[\sum_1^n\left|\frac{\partial u}{\partial x^i}\right|^2+|u|^2\right]dx.$$

(²) With the first definition u is much more precisely defined; its derivatives of the first two orders exist in the ordinary sense and are locally L^2.

(³) Recently the author was informed that P. Lax obtained a proof for the laplacian without the use of spherical harmonics.

where $r \leq 1$, $\alpha > 0$, u is a function of class C^2 vanishing outside of a compact contained in $0<|x|<r$, and c is a constant depending only on the dimension n.

Our proof is based on an inequality similar to (1.2) for an arbitrary operator A satisfying our restrictions. The next few sections will be taken up with the statement and proof of the inequality. (When A is the ordinary laplacian our line of proof becomes much simpler and in fact is shorter than the original proof of Heinz.)

2. We may and shall assume that the domain D is bounded and will consider it as a differentiable manifold of class C^∞. Consider there first the metric

$$ds^2 = a_{ij}\,dx^i\,dx^j \quad (^4)$$

where the matrix $\{a_{ij}\}$ is the inverse of $\{a^{ij}\}$.

For a fixed point $x_0 \in D$ and variable point $x \in D$ denote by r the geodesic distance from x_0 to x. With a constant ν to be defined in the next section we shall introduce new tensors

$$(2.1)\qquad \tilde{a}^{ij} = e^{2\nu r^2}a^{ij}, \qquad \tilde{a}_{ij} = e^{-2\nu r^2}a_{ij},$$

and the corresponding Laplace-Beltrami (L. B.) operator

$$(2.2)\qquad \tilde{\Lambda} = \frac{1}{\sqrt{\tilde{a}}}\frac{\partial}{\partial x^i}\sqrt{\tilde{a}}\,a^{ij}\frac{\partial}{\partial x^j}, \qquad \text{where}\quad \tilde{a} = \operatorname{Det}\{\tilde{a}_{ij}\}.$$

In the corresponding metric $d\tilde{s}^2 = \tilde{a}_{ij}\,dx^i\,dx^j$ let $\tilde{r}$ be the new geodesic distance between x_0 and x. It is clear that in both metrics ds^2 and $d\tilde{s}^2$ the geodesic lines issued from x_0 are the same and the distances r and $\tilde{r}$ are related as follows

$$(2.3)\qquad \tilde{r} = \int_0^r e^{-\nu\sigma^2}\,d\sigma.$$

Denote by $\tilde{S}_h(x_0)$ the geodesic sphere $\tilde{r} < h$ in the metric $d\tilde{s}^2$. Our basic inequality can be written as follows

$$(2.4)\qquad ch^2\int_{\tilde{S}_h(x_0)}\tilde{r}^{-2\alpha}|\tilde{\Lambda}u|^2\sqrt{\tilde{a}}\,dx \geq \int_{\tilde{S}_h(x_0)}\tilde{r}^{-2\alpha}\left[\tilde{a}^{ij}\frac{\partial u}{\partial x^i}\frac{\partial \bar{u}}{\partial x^j}+|u|^2\right]\sqrt{\tilde{a}}\,dx.$$

(⁴) From now on the usual convention will be applied that repeated indices imply summation.

Here u is a function of class C^∞ vanishing outside of a compact contained in $\tilde{S}_h(x_0) - \{x_0\}$, α any positive constant, $h \leqq r'$. r' and c are constants independent on u and α, depending only on the tensor a^{ij} and the position of x_0 in D.

In order to describe more specifically the dependence of the contants ν, r, and c on the tensor a^{ij} and the position of x_0, denote by 2 R the euclidean distance from x_0 to the boundary ∂D of D; by Θ the upper bound of the eigenvalues of the matrices $\{a^{ij}\}$ and $\{a_{ij}\}$ for all points x in the sphere $|x - x_0| \leqq$ R. Furthermore, let B_ε be the upper bound of absolute values of all derivatives of order $\leqq 3$ of the a^{ij} in the sphere $|x - x_0| \leqq R + \varepsilon$ and $B = \lim_{\varepsilon \nearrow 0} B_\varepsilon$. The third derivatives may exist only almost everywhere but will have a finite essential bound since the second derivatives are lipschizian. The possible discontinuities of third deriuatives are the reason for choosing $B = B_\varepsilon$ instead of $B = B_0$. We put

$$(2.5) \qquad \Gamma = \max\left(\frac{1}{R}, \Theta, B, n\right).$$

The constants ν, c, r', will then be determined as positive continuous functions of Γ, the first two increasing, the last decreasing. In addition r' will be determined so that the geodesic sphere $\tilde{S}_{r'}(x_0)$ be contained in the euclidean sphere $|x - x_0| < R$. We do not give here an exact determination of these functions since it is sufficient for our purposes to establish their existence without having the exact form.

We now remark that if formula (2.4) is established for tensors a^{ij} of class C^∞ with the constants ν, c, and r', satisfying the above specifications, then it is also valid for tensors of class $C^{(2,1)}$. In fact, by regularization of a^{ij} one obtains a sequence of tensors α^{ij} of class C^∞ whose constants Γ will converge to the constant Γ of a^{ij}. In addition, the α^{ij} and their derivatives of orders $\leqq 2$ will converge uniformly to the a^{ij} and their corresponding derivatives. It follows that in (2.4) every term depending on the tensor a^{ij} is a uniform limit of the corresponding terms for the tensors α^{ij} and hence the validity of the formula for α^{ij} implies its validity for a^{ij}. This fact will allow us in the next sections to restrict ourselves to a^{ij} of class C^∞.

3. Now let a^{ij} be a positive definite tensor of class C^∞ in D. For a neighborhood of x_0 we introduce the classical geodesic coordinate patch. To this effect consider $\mathcal{E}^n$ as the tangential space of our manifold D at x_0. The points (or vectors) in the tangential space will be denoted by ξ and its origin will be chosen at x_0. The norm in this space is given by the metric tensor a_{ij} at the point x_0 and by a suitable choice of coordinates ξ^i we can make it into a euclidean norm $|\xi|^2 = \Sigma(\xi^i)^2$. To a point ξ in this tangential space we assign a point x in D as follows : Take the geodesic line (in the metric $ds^2 = a_{ij}\, dx^i\, dx^j$) issued from x_0 and tangent to the vector ξ. On this geodesic line let x be the point at geodesic distance $|\xi|$ from x_0. The construction will be made only for $|\xi| \leqq r''$, where r'' is chosen small enough so that any two differents geodesics issued from x_0 and of length r'' have no points in common except x_0 and so that the geodesic sphere $S_{r''}(x_0)$ lies completely in the interior of the euclidean sphere $|x - x_0| < R$. By checking the classical proofs of existence and local uniqueness of geodesic lines it is easy to show that r'' can be chosen to depend only on Γ and to form a continuous decreasing function of Γ. The correspondence $\xi \to x$ is then a homeomorphism of class C^∞ transforming the sphere, $\Sigma_{r''}$: $|\xi| < r''$ onto $S_{r''}(x_0)$; clearly, $|\xi| =$ the geodesic distance r from x_0 to x.

We now choose a finite number of coordinate patches covering the hypersurface $\partial\Sigma_1 : |\xi| = 1$, with local coordinates $t^1, \ldots, t^{n-1}$. These coordinate patches will be fixed once and for all independently of the tensor a^{ij} (but depending, obviously, on the dimension n). In this way we obtain the geodesic polar coordinates $t^1, \ldots, t^{n-1}, r$ in the geodesic sphere $S_{r''}(x_0)$. For the metric ds^2 we now have two expressions

$$(3.1) \qquad ds^2 = g_{ij}\, d\xi^i\, d\xi^j = dr^2 + r^2 \gamma_{ij}\, dt^i\, dt^j,$$

where

$$g_{ij} = a_{k,l} \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^l}{\partial \xi^j}.$$

The γ_{ij} form a metric tensor on the hypersurface $\partial S_r(x_0)$ for every fixed $r > 0$. We write

$$\gamma = \mathrm{Det}\{\gamma_{ij}\}, \qquad \{\gamma^{ij}\} = \{\gamma_{ij}\}^{-1}.$$

By checking on the correspondence $\xi \to x$ and the expression of g_{ij} and γ_{ij} in terms of the tensor a_{ij}, the derivatives $\frac{\partial x^k}{\partial \xi^i}$ and the transformation of the cartesian coordinates ξ^i into polar coordinates t^i, r, we obtain easily that there exists a positive constant B' depending only on Γ, continuous and increasing as function of Γ, such that the eigenvalues of the matrices $\{\gamma_{ij}\}$ and $\{\gamma^{ij}\}$ as well as the values of γ and $\frac{1}{\gamma}$ on all hypersurfaces $\partial S_r(x_0)$ for $0 < r \leqq r''$ are bounded by B', and such that all the partial derivatives $\frac{\partial \gamma^{ij}}{\partial r}$, $\frac{\partial \gamma^{ij}}{\partial t^k}$, $\frac{\partial^2 \gamma^{ij}}{\partial r^2}$, $\frac{\partial^2 \gamma^{ij}}{\partial r\, \partial t^k}$, $\frac{\partial \gamma}{\partial r}$, $\frac{\partial \gamma}{\partial t^k}$, $\frac{\partial^2 \gamma}{\partial r^2}$, $\frac{\partial^2 \gamma}{\partial r\, \partial t^k}$, are bounded in absolute value by B'. Furthermore, if $r \to 0$, γ^{ij} converges uniformly considered as a tensor on $\partial \Sigma_1$ (γ_{ij} converges to the metric tensor induced on $\partial \Sigma_1$ by the euclidean metric).

From the well known property that $\frac{\partial g_{ij}(0)}{\partial \xi^k} = 0$ we get $\frac{\partial \gamma^{ij}}{\partial r} \to 0$ and $\frac{\partial \gamma}{\partial r} \to 0$ when $r \to 0$ and hence

(3.2) $$\left| \frac{\partial \gamma^{ij}}{\partial r} \right| \leqq B' r, \qquad \left| \frac{\partial \gamma}{\partial r} \right| \leqq B' r.$$

It is clear that $\frac{\partial \gamma^{ij}}{\partial r}$ is a contravariant tensor on the hypersurface $\partial S_r(x_0)$. The reason for changing the metric ds^2 into $d\tilde{s}^2 = e^{-2\nu r^2} ds^2$ is to make the last tensor positive definite for $0 < r \leqq r''$ (5). The geodesic polar coordinates for $d\tilde{s}^2$ are abtained from those for ds^2 only by the change of variable r into $\tilde{r}$ given by (2.3). The expression of $d\tilde{s}^2$ in terms of its polar coordinates is therefore

$$d\tilde{s}^2 = d\tilde{r}^2 + r^2 e^{-2\nu r^2} \gamma_{ij}\, dt^i\, dt^j = d\tilde{r}^2 + \tilde{r}^2 \tilde{\gamma}_{ij}\, dt^i\, dt^j.$$

(5) The author is indebted to Professor K. T. Smith for raising the conjecture that such a change of metric may achieve the desired result. The geometric significance of the change is that the new metric has a positive curvature in the neighborhood of x_0. If the metric ds^2 already had a non-negative curvature we would not have to make the change. This is actually what happens in the case treated by Heinz where ds^2 is the ordinary euclidean metric.

It follows that

$$\tilde{\gamma}_{ij} = \frac{r^2 e^{-2\nu r^2}}{\tilde{r}^2} \gamma_{ij} \quad \text{and, hence,} \quad \tilde{\gamma}^{ij} = \frac{\tilde{r}^2 e^{2\nu r^2}}{r^2} \gamma^{ij},$$

$$\frac{\partial \tilde{\gamma}^{ij}}{\partial \tilde{r}} = \frac{\tilde{r}^2 e^{3\nu r^2}}{\tilde{r}^2} \left[\frac{\partial \gamma^{ij}}{\partial r} + 2 \left(\frac{e^{-\nu r^2}}{\tilde{r}} - \frac{1}{r} + 2\nu r \right) \gamma^{ij} \right].$$

If t_i is any covariant tensor on $\partial S_r(x_0)$ we get from (3.2) and the properties of the constant B' that

$$\left| \frac{\partial \gamma^{ij}}{\partial r} t_i t_j \right| \leqq B' r \left(\sum_1^{n-1} |t_i| \right)^2 \leqq B'^2 (n-1) r \gamma^{ij} t_i t_j \leqq B'^2 \Gamma r \gamma^{ij} t_i t_j$$

and

$$\frac{\partial \tilde{\gamma}^{ij}}{\partial \tilde{r}} t_i t_j \geqq \frac{\tilde{r}^2 e^{3\nu r^2}}{r^2} \left[2 \left(\frac{e^{-\nu r^2}}{\tilde{r}} - \frac{1}{r} + 2\nu r \right) - B'^2 \Gamma r \right] \gamma^{ij} t_i t_j.$$

We choose $\nu = B'^2 \Gamma$. Since by (2.3) $\tilde{r} < r$ and since $e^{-\nu r^2} - 1 > -\nu r^2$ the square bracket in the last inequality is

$$> [2(-\nu r + 2\nu r) - B'^2 \Gamma r] = B'^2 \Gamma r.$$

Hence with this choice of ν the tensor $\frac{\partial \tilde{\gamma}^{ij}}{\partial r}$ is positive definite on all $\partial \tilde{S}_{\tilde{r}}(x_0)$ for

$$0 < \tilde{r} \leqq \tilde{r}'' = \int_0^{r''} e^{-\nu r^2}\, dr.$$

By a similar argument one verifies that the above choice of the constant ν implies $\frac{\partial \log \sqrt{\tilde{\gamma}}}{\partial r} < 0$.

It is immediately seen that for the metric and tensors with $\sim$ the same developments, properties, and evaluations hold as for those without $\sim$ except that the constant r'' should be replaced by $\tilde{r}''$ and B' by a constant $\tilde{B}'$ which is easily determined and which is again a continuous increasing function of Γ.

4. We come now to the proof of the basic inequality (2.4). Since, in the present section we deal only with the operators and tensors with $\sim$ and corresponding polar coordinates, it will simplify the notation to skip the $\sim$. The B-L operator A in the polar geodesic coordinates now has the following form

$$\text{(4.1)} \qquad Au = \frac{\partial^2 u}{\partial r^2} + \left(\frac{n-1}{r} + \frac{\partial \log \sqrt{\gamma}}{\partial r}\right)\frac{\partial u}{\partial r} + \frac{1}{r^2\sqrt{\gamma}}\frac{\partial}{\partial t^i}\sqrt{\gamma}\,\gamma^{ij}\frac{\partial u}{\partial t^j}.$$

It is important to notice that $\sqrt{\gamma}$ is the density corresponding to the metric $\gamma_{ij}\,dt^i\,dt^j$ on each hypersurface $\partial S_r(x_0)$ and hence depends on the choice of local coordinates t^i. But the expression $\frac{\partial \log \sqrt{\gamma}}{\partial r}$ is a function defined on $\partial S_r(x_0)$ and in view of (3.2) is bounded by $\frac{1}{2}B'^2 r$. The integral in the left member of the inequality (2.4) can now be written in the form

$$\text{(4.2)} \qquad I_h = \int_0^h \int_{\partial\Sigma_1} r^{-2\alpha}\,|Au|^2 r^{n-1}\sqrt{\gamma}\,dr\,dt.$$

This integral will be changed as follows. We introduce a new variable by putting $r = e^{-\rho}$. The function $u \equiv u(t, r)$ becomes now a function $v(t, \rho) = u(t, e^{-\rho})$, $t \in \partial\Sigma_1$ and $\chi < \rho < \infty$, $\chi = -\log h$. We next put $\beta = \alpha - \frac{n}{2} + 2$ and $v = e^{-\beta\rho} w$. Henceforward we will denote derivatives with respect to ρ by primes. After these transformations, our integral becomes

$$\text{(4.3)} \qquad I_h = \int_\chi^\infty \int_{\partial\Sigma_1} |w'' - (n-2+2\beta-\mu)w' + \beta(\beta+n-2-\mu)w + \Delta_\rho w|^2 \sqrt{\gamma}\,dt\,d\rho.$$

Here Δ_ρ denotes the B-L operator on the hypersurface $\partial S_r(x_0)$ for $r = e^{-\rho}$ with respect to the metric $\gamma_{ij}\,dt^i\,dt^j$, i. e. the operator $\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t^i}\sqrt{\gamma}\,\gamma^{ij}\frac{\partial}{\partial t^j}$. μ denotes the expression $\frac{\partial \log\sqrt{\gamma}}{\partial \rho}$. In view of our previous evaluation for $\frac{\partial \log \sqrt{\gamma}}{\partial r}$ and the fact that $\frac{\partial \log\sqrt{\gamma}}{\partial r} < 0$ we obtain

$$\text{(4.4)} \qquad 0 < \mu \leq \frac{1}{2} B'^2 e^{-2\rho}.$$

The calculations will be simplified if instead of the integral I_h we consider

$$\text{(4.5)} \qquad I_h^0 = \int_\chi^\infty \int_{\partial\Sigma_1} |w'' - (n-2+2\beta)w' + \beta(\beta+n-2)w + \Delta_\rho w|^2 \sqrt{\gamma}\,dt\,d\rho.$$

It is immediately seen that

$$\text{(4.6)} \qquad I_h \geq \frac{1}{2} I_h^0 - \int_\chi^\infty \int_{\partial\Sigma_1} \mu^2 |w' - \beta w|^2 \sqrt{\gamma}\,dt\,d\rho.$$

Our hypotheses concerning the function u imply that the function w is of class C^∞ and vanishes for ρ outside of a finite interval completely interior to the range $\chi < \rho < \infty$. We transform the integral I_h^0 by developing the expression $|\ldots|^2$ and using partial integration in two ways : one, integration with respect to ρ for fixed t and secondly, integration for fixed ρ over the hypersurface $\partial\Sigma_1$. This last integration by parts uses the well known formula that if τ^{ij} is a contravariant tensor on $\partial\Sigma_1$, and φ and ψ are two functions on $\partial\Sigma_1$, then

$$\int_{\partial\Sigma_1} \varphi \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial t^i}\sqrt{\gamma}\,\tau^{ij}\frac{\partial\overline{\psi}}{\partial t^j}\sqrt{\gamma}\,dt = -\int_{\partial\Sigma_1} \tau^{ij}\frac{\partial\varphi}{\partial t^i}\frac{\partial\overline{\psi}}{\partial t^j}\sqrt{\gamma}\,dt.$$

Combining these two procedures, one transforms the integral I_h^0 into the following form

$$\begin{aligned}
\text{(4.7)} \quad I_h^0 = &\int_\chi^\infty \int_{\partial\Sigma_1} \Big\{ |w''|^2 + (2\beta+n-2)\mu\,|w'|^2 \\
&\qquad - (2\beta+n-2+2\mu)(\gamma^{ij})\frac{\partial w}{\partial t^i}\frac{\partial \overline{w}}{\partial t^j} \\
&\qquad + \left|\Delta_\rho w + \left[\beta(\beta+n-2) + \left(\beta + \frac{n-2}{2}\right)\mu\right] w\right|^2 \Big\} \sqrt{\gamma}\,dt\,d\rho \\
&+ \int_\chi^\infty \int_{\partial\Sigma_1} \Big\{ 2\left[\left(\beta + \frac{n-2}{2}\right)^2 + \left(\frac{n-2}{2}\right)^2\right] |w'|^2 \\
&\qquad + 2\gamma^{ij}\frac{\partial w'}{\partial t^i}\frac{\partial \overline{w}'}{\partial t^j} \Big\} \sqrt{\gamma}\,dt\,d\varphi \\
&+ \int_\chi^\infty \int_{\partial\Sigma_1} \Big\{ \left(\beta + \frac{n-2}{2}\right) 2\,\mathrm{Re}\left[w\gamma^{ij}\frac{\partial \mu}{\partial t^i}\frac{\partial \overline{w}}{\partial t^j}\right] \\
&\qquad - [(\gamma^{ij})'' + (\mu' + \mu^2)\gamma^{ij}]\frac{\partial w}{\partial t^i}\frac{\partial \overline{w}}{\partial t^j} \\
&\qquad + \left[\beta(\beta+n-2)\mu' - \left(\frac{n-2}{2}\right)^2 \mu^2\right] |w|^2 \Big\} \sqrt{\gamma}\,dt\,d\rho \\
\equiv\ & I_h^{0,1} + I_h^{0,2} + I_h^{0,3}.
\end{aligned}$$

In the integral $I_h^{0,1}$ all the terms are positive since $\mu > 0$,

$$2\beta + n - 2 = 2\alpha + 2 > 2$$

and $-(\gamma^{ij})' = \frac{\partial \gamma^{ij}}{\partial r} e^{-\rho}$ is a positive definite matrix. It is easy to prove also that the integral $I_h^{0,3}$ is majorated as follows

$$|I_h^{0,3}| \leqq B'' e^{-\chi} I_h^{0,2}, \tag{4.8}$$

where B'' is again a constant depending only on Γ, continuously and increasingly. Similarly, for the integral in (4.6), we have

$$\int_\chi^\infty \int_{\partial\Sigma_1} \mu^2 |w' - \beta w|^2 \sqrt{\gamma}\, dt\, d\rho \leqq B'' e^{-\chi} I_h^{0,2}. \tag{4.9}$$

These facts are established by using the properties of the bound B' and also the following formulas and evaluations : formulas (3.2) and (4.4)

$$\mu' = (\log\sqrt{\gamma})'' = \frac{\partial^2 \log\sqrt{\gamma}}{\partial r^2} e^{-2\rho} + \frac{\partial \log\sqrt{\gamma}}{\partial r} e^{-\rho}, \qquad (\gamma^{ij})'' = \frac{\partial^2 \gamma^{ij}}{\partial r^2} e^{-2\rho} + \frac{\partial \gamma^{ij}}{\partial r} e^{-\rho},$$

$$\frac{\partial \mu}{\partial t^i} = -\frac{\partial^2 \log\sqrt{\gamma}}{\partial t^i\, \partial r} e^{-\rho},$$

$$\left| \int_\chi^\infty \int_{\partial\Sigma_1} \left(\beta + \frac{n-2}{2}\right) 2\,\mathrm{Re}\left[w \gamma^{ij} \frac{\partial \mu}{\partial t^i} \frac{\partial \bar{w}}{\partial t^j} \right] \sqrt{\gamma}\, dt\, d\rho \right|$$

$$\leqq \left(\beta + \frac{n-2}{2}\right)^2 \int_\chi^\infty \int_{\partial\Sigma_1} \left(\gamma^{ij} \frac{\partial \mu}{\partial t^i} \frac{\partial \mu}{\partial t^j}\right)^{\frac{1}{2}} |w|^2 \sqrt{\gamma}\, dt\, d\rho$$

$$+ \int_\chi^\infty \int_{\partial\Sigma_1} \left(\gamma^{ij} \frac{\partial \mu}{\partial t^i} \frac{\partial \mu}{\partial t^j}\right)^{\frac{1}{2}} \gamma^{ij} \frac{\partial w}{\partial t^i} \frac{\partial \bar{w}}{\partial t^j} \sqrt{\gamma}\, dt\, d\rho,$$

$$\gamma^{ij} \frac{\partial w'}{\partial t^i} \frac{\partial \bar{w}'}{\partial t^j} \geqq \frac{1}{B'} \sum_1^{n-1} \left| \frac{\partial w'}{\partial t^i} \right|^2.$$

Finally, we use the evalution

$$\int_\chi^\infty e^{-\tau\rho} |w|^2 \sqrt{\gamma}\, d\rho \leqq B' \frac{e^{-\tau\chi}}{\tau^2} \int_\chi^\infty |w'|^2 \sqrt{\gamma}\, d\rho \qquad (\tau > 0). \tag{4.10}$$

A similar formula is valid for w replaced by $\frac{\partial w}{\partial t^i}$. It follows now from (4.6), (4.8) and (4.9) that if $h = e^{-\chi} \leqq \frac{1}{6B'' + 1} < 1$.

$$I_h \geqq \frac{1}{4} I_h^{0,2} = \frac{1}{4} \int_\chi^\infty \int_{\partial\Sigma_1} \left\{ 2\left[\left(\beta + \frac{n-2}{2}\right)^2 + \left(\frac{n-2}{2}\right)^2\right] |w'|^2 + 2\gamma^{ij} \frac{\partial w'}{\partial t^i} \frac{\partial \bar{w}'}{\partial t^j} \right\} \sqrt{\gamma}\, dt\, d\rho. \tag{4.11}$$

We choose $r' = \frac{1}{6B'' + 1}$ and turn to the second member of the inequality (2.4). As in the first member we replace the coordinates x^k by geodesic polar coordinates, then introduce the variable $\rho\,(r = e^{-\rho})$ and also the function w. The integral becomes

$$\int_\chi^\infty \int_{\partial\Sigma_1} \left[e^{-2\rho}\left(|w'|^2 + \beta(\beta + \mu - 2)|w|^2 + \gamma^{ij} \frac{\partial w}{\partial t^i} \frac{\partial \bar{w}}{\partial t^j} \right) + e^{-4\rho} |w|^2 \right] \sqrt{\gamma}\, dt\, d\rho. \tag{4.12}$$

An argument similar to the one which led to the evaluations (4.8) and (4.9) [(4.10) being used with $\tau = 2$ instead of $\tau = 1$] gives for the integral (4.12) an upper bound $B''' e^{-2\chi} I_h^{0,2} = B''' h^2 I_h^{0,2}$, B''' depending again only on Γ, continuously and increasingly. Together with (4.11) this gives (2.4) with the constant $c = 4B'''$.

5. The remaining part of the proof of the theorem is relatively simple and we indicate it briefly.

If a function u satisfies the conditions of the theorem we notice first that obviously for some constant M' (depending only on M and Γ, continuously and increasingly) we have

$$|\tilde{\Lambda} u(x)|^2 \leqq M' \left[\tilde{a}^{ij} \frac{\partial u(x)}{\partial x_i} \frac{\partial \bar{u}(x)}{\partial x^j} + |u(x)|^2 \right] \qquad [x \in \tilde{S}_{r'}(x_0)]. \tag{5.1}$$

Let $r_0 = \frac{1}{3} \min\left(r', \sqrt{\frac{1}{2cM'}}\right)$. In (2.4) take $h = 3r_0$. Consider a function φ of class C^∞ vanishing outside of $\tilde{S}_{2r_0}(x_0)$ and $= 1$ in $\tilde{S}_{r_0}(x_0)$. Put $u_1 = \varphi u$. We first prove that the inequality (2.4) is valid with the function u_1 replacing u.

We note first that for each fixed $\alpha > 0$ the integrals in (2.4) are finite. To prove this, it is enough, in view of (5.1), to show that u_1 as well as all derivatives $\frac{\partial u_1}{\partial x^k}$ have zeros of infinite order at x_0 in the 2-mean. To prove this last fact we remark that by Sobolev's theorem [4] the second derivatives of u_1 being L^2, u is a function of class L^q with some $q > 2$. The Holder inequality then shows that u_1 having a zero of infinite order in 1-mean implies that it has such a zero in

2-mean. We then apply the identity

$$\int \sum_{1}^{n} \left| \frac{\partial v}{\partial x^k} \right|^2 dx = -\int v \Delta \bar{v} \, dx$$

(valid for v vanishing outside of a compact), to the function

$$v = \psi\left(\frac{x - x_0}{\varepsilon}\right) u_1.$$

Here : $\psi(x)$ is a function of class C^∞ vanishing for $|x| \geqq 1$, $= 1$ for $|x| \leqq \frac{1}{2}$ and $0 < \psi(x) < 1$ for $\frac{1}{2} < |x| < 1$; $0 < \varepsilon < 1$. Thus we obtain

$$\int_{|x-x_0| < \frac{\varepsilon}{2}} \sum \left| \frac{\partial u_1}{\partial x^k} \right|^2 dx \leqq \left(\int_{|x-x_0| < \varepsilon} |u_1|^2 dx \right)^{\frac{1}{2}} (C_0 + C_1 \varepsilon^{-1} + C_2 \varepsilon^{-2}),$$

where C_0, C_1, C_2, depend on u_1 but not on ε. This proves our statement about $\frac{\partial u_1}{\partial x^k}$.

For the above function ψ, put

$$K = \int_{|x|<1} \psi(x) \, dx \quad \text{and} \quad \psi_\varepsilon(x) = \frac{1}{K \varepsilon^n} \psi\left(\frac{x}{\varepsilon}\right).$$

Then by convoluting ψ_ε with the function equal to u_1 for $|x - x_0| \geqq 2\varepsilon$ and $= 0$ for $|x - x_0| < 2\varepsilon$ we obtain a function $u_{1,\varepsilon}$ satisfying, for sufficiently small ε, the requirements of inequality (2.4), and such that the integrals in (2.4) for $u_{1,\varepsilon}$ converge to the corresponding integrals for u_1 when $\varepsilon \to 0$.

The inequality (2.4) for u_1 and $h = 3r_0$ together with (5.1) and the properties : $u = u_1$ for $x \in \tilde{S}_{r_0}(x_0)$ and $\mathrm{ch}^2 M' \leqq \frac{1}{2}$ give, by an argument due essentially to T. Carleman (used in the form below by E. Heinz), that for all $\alpha > 0$.

$$\int_{\tilde{r} < r_0} \tilde{r}^{-2\alpha} \left[\tilde{a}^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \bar{u}}{\partial x^j} + |u|^2 \right] \sqrt{\tilde{a}} \, dx \leqq \int_{r_0 < \tilde{r} < 3r_0} \tilde{r}^{-2\alpha} \left[\tilde{a}^{ij} \frac{\partial u_1}{\partial x^i} \frac{\partial \bar{u}_1}{\partial x^j} + |u_1|^2 \right] \sqrt{\tilde{a}} \, dx$$

which implies that $u = 0$ in $\tilde{S}_{r_0}(x_0)$. Since r_0 is a positive continuous

and decreasing function of Γ which is uniformly bounded for x_0 in any domain D' completely interior to D, an obvious argument proves that $u = 0$ in D.

Remark 1. — For any θ with $0 \leqq \theta < 2$, we can obtain, instead of (2.4), the inequality

$$\frac{4\, \mathrm{ch}^{2-\theta}}{(2-\theta)^2} \int_{\tilde{S}_h(x_0)} \tilde{r}^{-2\alpha} |\tilde{A} u|^2 dx \geqq \int_{\tilde{S}_h(x_0)} \tilde{r}^{-2\alpha} \left[\tilde{r}^{-\theta} \tilde{a}^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \bar{u}}{\partial x^j} + \tilde{r}^{-\theta-2} |u|^2 \right] \sqrt{\tilde{a}} \, dx.$$

The argument leading to this inequality is exactly the same as for (2.4); but in evaluating the second member we use formula (4.10) with $\tau = 2 - \theta$ instead of $\tau = 2$. The above inequality allows us to obtain the unique continuation theorem when (1.1) is replaced by an inequality with a singularity at x_0

$$|A u(x)|^2 \leqq M \left\{ |x - x_0|^{-\theta} \sum_{1}^{n} \left| \frac{\partial u(x)}{\partial x^i} \right|^2 + |x - x_0|^{-\theta-2} |u(x)|^2 \right\}.$$

It is interesting to notice that if we consider solutions of the last inequality with $\theta > 2$, simple counter-examples show that the unique continuation theorem is no longer valid.

Remark 2. — We mention that our theorem allows us to establish the uniqueness of elliptic solutions of the Cauchy problem for general linear, quasi-linear, and for certain types of non-linear partial differential equations of second order. As an example of such a non-linear equation let us consider

$$\mathrm{Det} \left\{ \frac{\partial^2 u(x)}{\partial x^i \partial x^j} \right\} = F\left[x, u(x), \frac{\partial u(x)}{\partial x^1}, \ldots, \frac{\partial u(x)}{\partial x^n} \right].$$

The ellipticity of a solution $u(x)$ means here essentially that $u(x)$ is a convex or concave function; more precisely that the matrix $\left\{ \frac{\partial^2 u(x)}{\partial x^i \partial x^j} \right\}$ is positive or negative definite for each x in $\bar{D}$. One proves here the following statement :

If two such elliptic solutions $u(x)$ and $v(x)$ are of class $C^{4,1}$ in $\bar{D}$ and have the same Cauchy data $u = v$ and $\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n}$ on a portion of the boundary of $\bar{D}$ of class $C^{4,1}$ then $u(x) = v(x)$ everywhere in D.

The function F is supposed here continuous in all its arguments and lipschitzian in all except possibly the first.

Remark 3. — Our main theorem can be immediately extended to systems $(u_1, \ldots, u_m)$ of m functions satisfying a system of m inequalities of the form

$$|Au_k|^2 \leqq M \sum_{l=1}^{m} \left(\sum_{i=1}^{n} \left| \frac{\partial u_l}{\partial x^i} \right|^2 + |u_l|^2 \right) \qquad (k = 1, \ldots, m),$$

with an operator A (the same for each u_k) of the same type as before.

As corollary we obtain a unique continuation theorem for harmonic exterior differential forms $h_{i_1 \ldots i_p} dx^{i_1} \ldots dx^{i_p}$ on a Riemannian manifold with a metric $g_{ij} dx^i dx^j$ since in each system of local coordinates the components satisfy the equations

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \sqrt{g}\, g^{kl} \frac{\partial h_{i_1 \ldots i_p}}{\partial x^l} = L_{i_1 \ldots i_p}^{j_1 \ldots j_p} h_{j_1 \ldots j_p},$$

where $L_{i_1 \ldots i_p}^{j_1 \ldots j_p}$ are all linear differential operators of orders $\leqq 1$.

Addendum. — Quite recently Dr. H. O. Cordes, sent us his manuscript where he proves the general theorem. His proof also relies on an extension of Heinz' inequality and in order to obtain it he also uses the basic idea of multiplying the original operator A by a correcting factor. The form of the proof is however quite different from ours and it allows him to weaken the restriction on a^{ij} from $C^{2,1}$ to C^2. It seems probable that a more thorough analysis of the dependence of $\frac{\partial \gamma^{ij}}{\partial r}$ on the a^{ij} would make it possible to get rid of the dependence of our bounds on third derivatives of a^{ij} and thus weaken the restriction on a^{ij} to $C^{1,1}$. Any further weakening of these restrictions would seem to imply radical changes in the proof since we would loose then the local uniqueness of geodesics.

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