Combinatorial Argument

Tuesday, September 10, 2019 3:14 PM

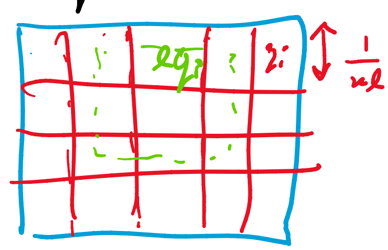
From Logunov - Malinnikova,

Lemma 4.1

Lemma Let $h \in L^2(\Omega)$.

Consider a cube $Q \subset \Omega$ partitioned into $(K\ell)^d$ equal cubes $\{q_i\}$.

w/ side length $\frac{1}{K\ell}$



($\ell > 2\sqrt{d}$ is an odd integer
K is just a number).

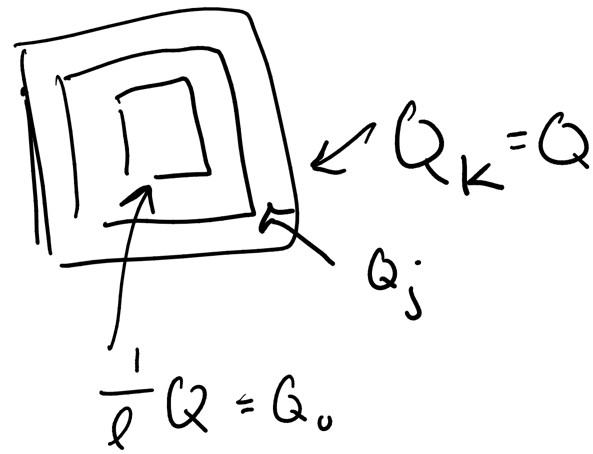
Let $N_{min} := \min\{N(h, q_i) \mid \ell q_i \subset Q\}$.

and assume $N_{min} \geq \frac{2d\ln(\ell)}{\ln(2)} \gtrsim d$.

Then $N(h, \frac{1}{2}Q) \geq \frac{1}{2}K N_{min}$

(Of course since it is imperative to control growth of N as $\operatorname{diam}(Q) \to 0$, this lemma says that you can control the minimum N.)

pf.



By def,
$$\frac{\int_{\ell q_i} |h|^2}{\int_{q_i} |h|^2} \geq 2^{N_{min}}$$

$$\Rightarrow 2^{N_{min}} \int_{Q_j} |h|^2 = 2^{N_{min}} \sum_{q_i \subset Q_j} \int_{q_i} |h|^2$$

$$\leq \sum_{q_i \subset Q_j} \int_{\ell q_i} |h|^2 \quad \left(\begin{array}{l} q_i \subset Q_j \\ \Rightarrow \ell q_i \subset Q_{j+1} \end{array} \right)$$

$$\leq \ell^d \int_{Q_{j+1}} |h|^2 .$$

also $2^{N_{min}} \geq \ell^{2d}$ by assumption.

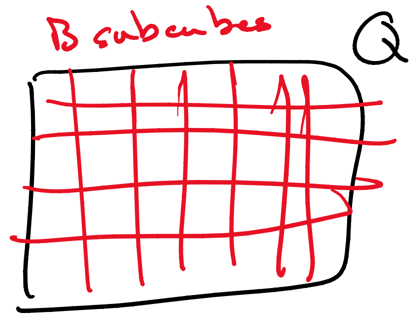
$$\Rightarrow 2^{N_{min}/2} \int_{Q_j} |h|^2 \leq \int_{Q_{j+1}} |h|^2$$

$$\Rightarrow 2^{K N_{min}/2} \int_{\frac{1}{\ell} Q} |h|^2 \leq \int_Q |h|^2 .$$

This next lemma extrapolates the previous lemma to a large-scale, off-center doubling estimate. Also, we possibly have something to say about the subcubes that are near the boundary.

Lemma (Lemma 4.2)

Let u satisfy $Lu=0$ in Q. There exists constants $B_0 = B_0(d,L)$ and $\delta = \delta(d) > 0$ s.t. if Q is partitioned into $B^d > B_0$ equal subcubes



then at least half of these subcubes, q, satisfy

$$\sup\{N(u,q') \mid q' \subset q\} \le \max\left(2d \frac{\ln L}{\ln 2}, \frac{\sup\{N(u,Q') \mid Q' \subset Q\}}{B^\delta}\right)$$

Note that trivially we have that

$$\sup\{N(u,q') \mid q' \subset q\} =: \tilde{N}(u,q) \le \tilde{N}(u,Q)$$

$$\sup\{N(h,q') \mid q' \subset q\} =: \tilde{N}(h,q) \leq \tilde{N}(h,Q)$$

because $q \subset Q$.

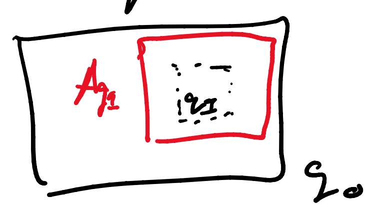
We gain a factor of B^δ with this lemma.

Q: What does $\delta(n)$ look like?

Pf:

Let $\tilde{N}(h,Q) =: N_0$

Recall: Monotonicity property of harmonic function (a manifold $\subset \mathbb{R}^n$)
$\exists\, A = A(d) \in \mathbb{Z}_+$, $C_0 = C_0(d) > 1$, $\rho = \rho(M) > 0$ s.t. if $A q_1 \subset q_0$
q_1 and q_0 are cubes with



then $N(h,q_1) \leq C_0 N(h,q_0)$

Let $K > 4C_0$

Divide Q into $(\ell K)^d$ subcubes

By the previous lemma, at least one subcube, q, satisfies

$$N(h,q) \leq \frac{2}{\ell} N(h, \tfrac{1}{\ell}Q)$$

$$N(h,q) \le \frac{2}{K} N\left(h, \tfrac{1}{\ell} Q\right)$$

$$\le \frac{2}{K} \tilde{N}(h,Q) = \frac{2}{K} N_0$$

(if N_0 is large enough).

By monotonicity condition,

$$\tilde{N}\left(h, \tfrac{1}{A} q\right) \le C_0 N(h,q) \le \frac{2}{K} C_0 N_0$$

$$\le \frac{1}{2} N_0 \text{ by assumption of } K > 4C_0$$

$\Rightarrow$ If we divide Q into $(\ell K A)^d$ subcubes, then at least one, c_0, will satisfy

$$\tilde{N}(h, c_0) \le \frac{1}{2} N_0 ,$$

Let $M := (\ell K A)^d$

all others cubes, c_i, will trivially have

$$\tilde{N}(h, c_i) \le N_0 = \tilde{N}(h,Q)$$

Now subdivide each of c_i into $(\ell K A)^d$ subcubes, c_{ji}, then one, c_{0i}, will satisfy

$$\tilde{N}(h, c_{0i}) \le \frac{1}{2} \tilde{N}(h, c_i) \le \frac{1}{2} \tilde{N}(h,Q)$$

$$\tilde{N}(h, c_{00}) \le \frac{1}{2} \tilde{N}(h, c_0) \le \frac{1}{4} \tilde{N}(h,Q).$$

By induction

At jth step,

$\exists$ cube q w/ $\tilde{N}(h,q) \le 2^{-j} N_0$

1 cube q w/ $N(h,q) \leq 2^{-j} N_0$

$\binom{j}{1}(M-1)$ cubes q w/ $\tilde{N}(h,q) \leq 2^{-(j-1)} N_0$

$\binom{j}{2}(M-1)^2$ cubes q w/ $\tilde{N}(h,q) \leq 2^{-(j-2)} N_0$

⋮

$\binom{j}{m}(M-1)^m$ cubes q w/ $\tilde{N}(h,q) \leq 2^{-(j-m)} N_0$
$\left(\text{If } N_0 2^{-j} \geq \frac{2d \ln R}{\ln 2} \right)$.

Let $X_1, \ldots, X_j$ be i.i.d. random variables such that $\mathbb{P}(X_1 = 1) = \frac{1}{M}$
and $\mathbb{P}(X_1 = 0) = (M-1)\frac{1}{M}$

By Law of Large numbers.

$$\mathbb{P}\left(\frac{\sum_{i=1}^{j} X_i}{j} > \frac{1}{2M} \right) \to 1 \text{ as } j \to \infty.$$

$\Rightarrow$ For j big enough

$$\frac{1}{2} \leq \mathbb{P}\left(\sum^{j} X_i \geq \frac{j}{2M} \right) = \sum_{j \geq k \geq \frac{j}{2M}} \mathbb{P}\left(\sum X_i = k \right)$$

$$= \sum_{k \geq \frac{j}{2M}} \binom{j}{k} \frac{(M-1)^{j-k}}{M^j}$$

$$= \sum_{i(2M)-j} \binom{j}{k} (M-1)^k$$

$$= \frac{\sum_{k \leq j\frac{(2M)-j}{2M}} \binom{j}{k} (M-1)}{M^j}$$

$\Rightarrow$ at least half cubes, q, at step j

have $\tilde{N}(h,q) \leq \tilde{N}(h,Q)\, 2^{-(j - \frac{(2M-1)j}{2M})}$

$$\leq \tilde{N}(h,Q) \boxed{2^{-\frac{j}{2M}}}$$

$$\shortparallel$$

$$B^{-\delta}$$