Density of Zeroes

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Lemma (Lemma 8.1 from "Lower bound" Logunov)

Let $B \subset M$ be a geodesic ball.
$\exists\ C_0 = C_0(M,B) > 0,\ \lambda_0 = \lambda_0(M,B) > 0$ s.t.
if $\lambda > \lambda_0$ then

$$Z_\lambda \text{ is } \frac{C_1}{\sqrt{\lambda}} \text{ dense in } B.$$

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pf:

Define the function

$$h(x,t) := u_\lambda(x)\, e^{\sqrt{\lambda}\, t} \quad \text{on } B \times \mathbb{R}.$$
$$(\Delta h = 0)$$

Let $y \in B \times [-1,1]$

Harnack ineq. says that $\exists C > 1$
$r > 0$ s.t. if $0 < r < r_0$, $h > 0$
on $B_{M\times\mathbb{R}}(y,r)$ then for $z \in B_{M\times\mathbb{R}}(y, r/2)$

$h(z) < C\, h(y)$

$\Rightarrow$ conversely that

if $|h(z)| > C\,|h(y)|$ then

$\Rightarrow$ conversely

if $|h(z)| \geq C\,|h(y)|$, then

h changes sign in $B_{M\times\mathbb{R}}(y, r)$

Let $C_1 = \log C$

Suppose $y = (x, 0)$ and $z = (x, \frac{C_1}{\sqrt{\lambda}})$

Then $h(z) = e^{C_1} h(y) = C h(y)$.

$\Rightarrow$ if λ is big enough

since $z \in B_{M\times\mathbb{R}}(y, \frac{3}{2}\frac{C}{\sqrt{\lambda}})$

$\exists\ \omega \in B_{M\times\mathbb{R}}(y, 3\frac{C}{\sqrt{\lambda}})$ s.t.

$h(\omega) = 0$

$\{h = 0\} = E_\lambda \times \mathbb{R}$ and $d_{M\times\mathbb{R}}(\omega, y) < 3\frac{C}{\sqrt{\lambda}}$

$\Rightarrow\ d_M(\hat{\omega}, x) < 3\frac{C}{\sqrt{\lambda}}$ $\square$.