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The effect of curvature on convexity properties of harmonic functions and eigenfunctions

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Dedicated to Shmuel Agmon with admiration and gratitude on the occasion of his 90th birthday

Abstract

We give a proof of the growth bound of Laplace–Beltrami eigenfunctions due to Donnelly and Fefferman which is probably the easiest and the most elementary one. Our proof also gives new quantitative geometric estimates in terms of curvature bounds which improve and simplify previous work by Garofalo and Lin. The proof is based on a generalization of a convexity property of harmonic functions in $\mathbb{R}^n$ to harmonic functions on Riemannian manifolds following Agmon's ideas.

1. *Introduction*

In their seminal paper [6], Donnelly and Fefferman found growth bounds (DF-growth bound) for eigenfunctions on compact Riemannian manifolds. Roughly, they showed that a λ-eigenfunction grows like a polynomial of order $\sqrt{\lambda}$ at most. This result is central in the study of eigenfunctions. In [6], it was applied to prove Yau's conjecture on real analytic manifolds. Namely, sharp upper and lower bounds on the size of the nodal set on real analytic manifolds were found. The proof of the growth bound in [6] went through a fine version of a Carleman-type inequality for the operator $\Delta + \lambda$, with a careful geometric choice of the weight function.

Soon after, Lin [12], based on an earlier work with Garofalo [7], gave a simpler proof of the growth bound. This proof is based on properties of the spherical L^2-norm, $q(r)$ (defined in (2.1)), for harmonic functions. It had been known [1, 3] that in $\mathbb{R}^n$, $\log q$ is monotonically increasing and convex as a function of $\log r$. Equivalently, $rq'(r)/q(r)$ is monotonically increasing. Garofalo–Lin showed that for a harmonic function defined on a general Riemannian manifold $e^{\Lambda r} rq'/q$ is monotonically increasing in (0, R), where Λ and R are some positive constants depending on bounds on the Riemannian metric, on its first derivatives and on the ellipticity constant of the Riemannian metric. This result can be viewed as an approximated convexity result. The proof of this result was based on a non-trivial geometric variational argument which was first used by Almgren [3].

The first aim of this paper is to give new geometric estimates on Λ and R in terms of the curvature of the manifold. Namely, we find that all one needs is a lower and an upper bound on the sectional curvature in order to guarantee the existence of Λ and R. Moreover, we show that in fact $e^{C_1 r^2 K} rq'(r)/q(r)$ is monotonic in $(0, R)$, where K is an upper bound on the curvature, R is the minimum of $C_2/\sqrt{K^+}$ and the injectivity radius, and C_1, C_2 depend only on the dimension of the manifold. We emphasize that our result distinguishes between negative and positive curvatures. This is the content of the main Theorem 2.2.

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The second aim of this paper is to have a simple proof of the DF-growth bound for eigenfunctions. Owing to the importance of this result, three simplifications to its proof had previously been given by different authors in the course of the years, which we briefly survey.

The idea of Lin [9, 12] was to consider a conic manifold, N, over M and to extend the eigenfunction u_λ to a harmonic function on N. Then, Lin applied the monotonicity property of $e^{\Lambda r} r q'/q$ from [7] to the harmonic function obtained, and went back to the eigenfunction.

In [10], Jerison and Lebeau applied a similar extension of eigenfunctions. Then, they could use standard Carleman-type inequalities for harmonic functions, instead of the original approach taken by Donnelly and Fefferman in which a special and delicate Carleman-type inequality for eigenfunctions was used.

In two dimensions, Nazarov–Polterovich–Sodin [13] took advantage of the conformal coordinates, thus allowing them to simplify the problem by considering only the standard Laplace operator in $\mathbb{R}^2$. Then, they extend the eigenfunction to a harmonic function on $N = M \times \mathbb{R}$, and apply the convexity argument to the harmonic function (in $\mathbb{R}^3$) obtained. Their proof of the convexity of $\log q$ is considerably simpler than the variational approach taken in [7]. It is close in spirit to Agmon's approach. This gives the easiest proof of the DF-growth bound in two dimensions, since there is no need for variational arguments or Carleman-type inequalities at all.

This paper extends the work started in [13], to dimensions greater than or equal to three, where no conformal coordinates exist. We follow and generalize Agmon's ideas in [1], where a general approximated convexity theorem for second-order elliptic equations is proved by considering them as an abstract second-order ordinary differential equation. Our contribution here comes in adding the geometric point of view, clarifying the way curvature affects the Euclidean result. Our proof also simplifies and improves Agmon's results in [1]. In this way, we are able to circumvent the need to use the non-trivial variational argument in [7] or any Carleman-type inequality.

Organization of the paper. The main result is presented in Section 2. In Section 3, we recall a way that eigenfunctions can be extended to harmonic functions and the translation of the convexity property of harmonic functions to a local growth bound on eigenfunctions. In Section 4, we conclude the proof of the DF-growth bound on compact manifolds and we outline the proof of Yau's conjecture in [6]. Sections 3 and 4 are strongly based on [13]. In Section 5, we give the proof of the main theorem. In Section 6, we consider constant curvature manifolds as examples to the main theorem and find a second proof in some of these cases. In Section 7, we discuss several open questions.

Notation. Throughout this paper C_i and $C_i(n)$ denote positive constants which depend only on dimension. The positive constants $C_g(\ldots)$ depend on bounds on the metric g, its first derivatives, its ellipticity constant and additional parameters appearing in parentheses.

2. *Main Theorem: a perturbed* log*-convexity property of harmonic functions*

Let u be a harmonic function in $\mathbb{R}^n$. Let $q(r)$ denote the square of the spherical L^2-norm:

$$q(r) := \int_{S_r} u^2 \, d\sigma_r,$$

where S_r denotes the sphere of radius r centred at 0, and $d\sigma_r$ is the standard area measure on S_r. It is easy to check that q is a convex function of $\log r$. It turns out that even $\log q$ is a convex function of $\log r$.

THEOREM 2.1 [1]. *The function q has the following two properties*:

(i) $(\log q)'(r) \geqslant (n-1)/r$;
(ii) $(\log q)''(r) + (1/r)(\log q)'(r) \geqslant 0$.

In two dimensions, this can be seen by a complex analysis argument. In higher dimensions, this fact goes back at least to Agmon [**1**], and it was rediscovered by Almgren [**3**]. Landis [**11**, Chapter II.2] also found several results close in spirit to that one. All these kinds of results were inspired by Hadamard's three circles theorem (see [**2**, Chapter 6.2]), which shows log-convexity of the spherical L^∞-norm for a holomorphic function.

Remark. It is somewhat surprising that the fundamental solution does not play a role here: $\log q$ is a convex function of $\log r$ in *all* dimensions. The weaker statement is that $\log q$ is a convex function of $G(r) = -1/r^{n-2}$, which is equivalent to $(\log q)''(r) + \frac{n-1}{r}(\log q)'(r) \geqslant 0$.

When considering harmonic functions on manifolds, one expects a perturbed version of Theorem 2.1 in small geodesic balls. However, it is not clear *a priori* how far from the centre this perturbation goes and how curvature controls it. Theorem 2.2 will give an answer to these questions. Let u be a harmonic function defined in a small geodesic ball of a Riemannian manifold N. Let

$$q(r) := \int_{S(r)} u^2 \, dA_r, \tag{2.1}$$

where $S(r)$ is a geodesic sphere centred at $p \in N$, and dA_r is the area measure on $S(r)$. We let Sec_N denote the sectional curvature of N, $K^+ = \max\{K, 0\}$, $\Theta(K)$ equal one if $K > 0$ and zero if $K \leqslant 0$,

$$\sin_K r = \begin{cases} \dfrac{\sin(r\sqrt{K})}{\sqrt{K}}, & K > 0, \\ r, & K = 0, \\ \dfrac{\sinh(r\sqrt{-K})}{\sqrt{-K}}, & K < 0, \end{cases}$$

and $\cot_K r = (\sin_K r)'/(\sin_K r)$. We can now state our main result.

Theorem 2.2. *Let N be a Riemannian manifold of n dimensions. Let u be a harmonic function on a geodesic ball in N, and q be defined as in (2.1). Let $\kappa, K \in \mathbb{R}$, $\kappa \leqslant K$. Let $R = \min(\mathrm{inj}(M), \pi/(2\sqrt{K^+}))$. We have the following statements.*

(i) *If $\mathrm{Sec}_N \leqslant K$, then $q(r)/(\sin_K r)^{n-1}$ is monotonically increasing for $r < R$. Equivalently,*

$$(\log q)'(r) \geqslant (n-1)(\cot_K r).$$

(ii) *If $\kappa \leqslant \mathrm{Sec}_N \leqslant K$, then for $r < R$,*

$$\begin{aligned} &(\log q)''(r) + (\cot_K r)(\log q)'(r) + (n+1)(\cot_\kappa r - \cot_K r)(\log q)'(r) \\ &\quad \geqslant -K - (n-2)K^+ - (n-1)(K-\kappa) - \frac{n}{2}(n-1)(n-2)(K-\kappa)\Theta(K). \end{aligned}$$

The proof of the theorem is given in Section 5.

Remarks.

(1) The result shows that in the case where $K < 0$ and where $K - \kappa$ is small we have actual convexity, since the right-hand side in Theorem 2.2(ii) is positive. One could state part (ii) of the theorem with the function $\tilde{q} = q/(\sin_K r)^{n-1}$ replacing q: then, the right-hand side becomes $-(n-2)K^- - c_1(n)(K-\kappa)$ which gives 'advantage' to positive curvature (see also the discussion in Section 7.2).

(2) For the constant non-positive curvature case, we obtain an exact convexity statement for $\log q$.

(3) Comparing with the result of Garofalo and Lin [7], from Theorem 2.2(ii), one deduces that $e^{c_2(n)r^2K}rq'(r)/q(r)$ is monotonically increasing for $r < R$. Observe that besides the explicit estimates of Λ and R mentioned in the introduction this also gives a correction of the result in [7] in the power of r in the exponential term. Moreover, the statement here is more geometric in nature.

We would now like to have an integrated version of Theorem 2.2. We restrict our attention to the case $\kappa = -K$, $K > 0$. We obtain a local doubling estimate for harmonic functions (see proof in Section 5.5).

Corollary 2.3. *Let N be a complete Riemannian manifold of dimension n with $|\mathrm{Sec}_N| \leqslant K$. Then*

$$\frac{q(2r)}{q(r)} \leqslant \left(\frac{q(2s)}{q(s)}\right)^{1+c_3(n)r^2K}$$

for all $r < s < 1/(4\sqrt{nK})$.

3. *Harmonic extension of eigenfunctions*

In this section, we recall a connection between harmonic functions and eigenfunctions found in [10, 12, 13]. Let M be a Riemannian manifold of dimension m. Let u_λ be a λ-eigenfunction on M. Consider the direct product Riemannian manifold $N = M \times \mathbb{R}$ of dimension $n = m + 1$, where the metric on $\mathbb{R}$ is the standard one. Let H be the following function on N:

$$\forall x \in M,\ t \in \mathbb{R}, \quad H(x,t) := u_\lambda(x)\cosh(\sqrt{\lambda}t).$$

Observe that H extends u_λ to N and is harmonic on N, since the Laplacian on N can be written as

$$\Delta_N u = \Delta_M u + \frac{\partial^2 u}{\partial t^2}.$$

On N, we take geodesic coordinates $(r, \theta_1, \ldots, \theta_{n-1})$ in a neighbourhood of the point $(p, 0) \in N$. In these coordinates, the metric g_N takes the following form:

$$g_N = dr^2 + r^2 a_{ij}\, d\theta^i\, d\theta^j, \quad 1 \leqslant i,j \leqslant n-1.$$

We let $\hat{\theta} = (\theta_1, \ldots, \theta_{m-1})$, and $b_{ij}(r, \hat{\theta}) := a_{ij}(r, \hat{\theta}, 0)$, then

$$g_M = dr^2 + r^2 b_{ij}\, d\theta^i\, d\theta^j, \quad 1 \leqslant i,j \leqslant m-1.$$

Accordingly, the equation $\Delta_N H = 0$ can be written in these coordinates as

$$H_{rr} + \left(\frac{n-1}{r} + \gamma(r,\theta)\right) H_r + \frac{1}{r^2}\Delta_{S(r)} H = 0,$$

where $\gamma(r,\theta) = (\sqrt{a})_r/\sqrt{a}$ with $a = \det(a_{ij})$, and $\Delta_{S(r)}$ is the spherical Laplacian on the geodesic sphere of radius r:

$$\Delta_{S(r)} H := \frac{1}{\sqrt{a}} \frac{\partial}{\partial \theta^i} \left(\sqrt{a} a^{ij} \frac{\partial H}{\partial \theta^j} \right).$$

The following lemma relates $q(r)^{1/2}$, the spherical L^2-norm of the harmonic function H on an $(n-1)$-dimensional sphere of radius r, to $M_r(u_\lambda)$, the L^∞-norm of the eigenfunction u_λ on an $m = (n-1)$-dimensional *ball* of radius r. Let $M_r(u_\lambda) := \max_{B(p,r)} |u_\lambda(x)|$.

Lemma 3.1. *Suppose M is a complete Riemannian manifold with bounded geometry. Fix $0 < \alpha < 1$. Then, for all $0 < r < \mathrm{inj}_M$,*

$$C_\alpha r^m (1 + r\sqrt{\lambda})^{-m} M_{\alpha r}(u_\lambda)^2 \leqslant q(r) \leqslant C_2 r^m e^{2r\sqrt{\lambda}} M_r(u_\lambda)^2,$$

where C_α depends both on α and the metric, and C_2 depends on the metric alone.

Proof. Let us denote by $d\sigma(\hat{\theta})$ the standard volume form on the unit sphere of dimension $m-1$. Then

$$\begin{aligned} q(r) &= 2\int_0^r \int u_\lambda(\rho,\hat{\theta})^2 \cosh^2(\sqrt{\lambda}\sqrt{r^2-\rho^2}) \cdot \rho^{m-1} \sqrt{b(\rho,\hat{\theta})} \frac{r}{\sqrt{r^2-\rho^2}}\, d\hat{\theta}\, d\rho \\ &\leqslant C M_r(u_\lambda)^2 (3 + e^{2r\sqrt{\lambda}}) \int_0^r \int_{S^{m-1}} \rho^{m-1} \frac{r}{\sqrt{r^2-\rho^2}}\, d\sigma(\hat{\theta}) d\rho \\ &= C\omega_m r^m M_r(u_\lambda)^2 (3 + e^{2r\sqrt{\lambda}}), \end{aligned}$$

where we used the fact that the volume element is bounded from above by the metric [**4**, Chapter 11, Theorem 15].

On the other hand, we have

$$q(r) \geqslant 2\int_0^r \int u_\lambda(\rho,\hat{\theta})^2 \rho^{m-1} \sqrt{b}\, d\hat{\theta}\, d\rho = \int_{B^m(p,r)} u_\lambda^2 \; d\,\mathrm{Vol}_M.$$

Hence, from elliptic regularity we get

$$q(r) \geqslant C_\alpha M_{\alpha r}(u_\lambda)^2 r^m (1 + r\sqrt{\lambda})^{-m},$$

where C_α depends on the metric and on α. □

From Corollary 2.3 and Lemma 3.1 we obtain the following theorem.

Theorem 3.2. *Let M be a complete Riemannian manifold of dimension m with $|\mathrm{Sec}_M| \leqslant K$. Then for all $r \leqslant s < C/\sqrt{K}$,*

$$\frac{M_{3r}(u_\lambda)}{M_{2r}(u_\lambda)} \leqslant C_1 e^{C_2\, s\sqrt{\lambda}} \left(\frac{M_{8s}(u_\lambda)}{M_{3s}(u_\lambda)} \right)^{1 + C_3 r^2 K},$$

where the constants C_2 and C_3 denote positive constants which depend only on the injectivity radius of M, while C_1 depends on bounds on the metric, its derivatives and its ellipticity constant.

Remark. The subindices $3r, 2r, 8s$, and $3s$ can be replaced by $\beta r, r, \gamma s$, and s, respectively, where $1 < \beta < 2$ and $\gamma > \beta$. The constants C_2 and C_3 can be taken to be independent of β and γ, while $C_1 \to \infty$ as $\gamma/\beta \to 1$.

4. *Two global growth estimates*

In this section, we deduce from the local inequality in Theorem 3.2 two global results in the compact case.

4.1. *Large values on large balls*

THEOREM 4.1. *Let M be a compact Riemannian manifold of dimension m. Then for all eigenfunctions u_λ and $r>0$,*

$$\frac{\max_{B(x,r)}|u_\lambda|}{\max_M|u_\lambda|}\geqslant C_g(r,d_M)e^{-C_2 d_M\sqrt{\lambda}}\quad\forall x\in M,$$

where d_M is the diameter of M.

Proof. Normalize u_λ so $\max_M|u_\lambda|=1$. Take $r=s$ in Theorem 3.2. We obtain

$$M_{3r}(u_\lambda)^{2+C_3r^2K}\leqslant C_1e^{C_2r\sqrt{\lambda}}M_{8r}(u_\lambda)^{1+C_3r^2K}M_{2r}(u_\lambda)\leqslant C_1e^{C_2r\sqrt{\lambda}}M_{2r}(u_\lambda). \tag{4.1}$$

Let $|u_\lambda(x_0)|=1$. Fix $r_0>0$ small enough in order to apply Theorem 3.2. Take a point x in M. There exists a sequence of points $x_0,x_1,\dots,x_N=x$ such that $d(x_k,x_{k+1})<r_0$ for $0\leqslant k\leqslant N-1$, where N only depends on r_0 and the diameter of M. Inequality (4.1) gives

$$\begin{aligned}\max_{B(x_k,2r_0)}|u_\lambda|&\geqslant C_1^{-1}e^{-C_2r_0\sqrt{\lambda}}\max_{B(x_k,3r_0)}|u_\lambda|^{2+C_2r^2K}\\&\geqslant C_1^{-1}e^{-C_2r_0\sqrt{\lambda}}\max_{B(x_{k-1},2r_0)}|u_\lambda|^3.\end{aligned}\tag{4.2}$$

Multiplying the inequalities (4.2) for $1\leqslant k\leqslant N$ gives

$$\max_{B(x,2r_0)}|u_\lambda|\geqslant C_1^{-N}e^{-C_2Nr_0\sqrt{\lambda}}\geqslant C_1^{-N}e^{-C_2d\sqrt{\lambda}}.\qquad\square$$

4.2. *Global DF growth bound*

THEOREM 4.2 [6]. *For all eigenfunctions u_λ, $x\in M$ and $r>0$,*

$$\frac{\max_{B(x,3r)}|u_\lambda|}{\max_{B(x,2r)}|u_\lambda|}\leqslant C_g(d_M)e^{C_2d_M\sqrt{\lambda}}.$$

Proof. Let $R>0$ be as in Theorem 4.1. If $r\geqslant R$, the theorem follows from Theorem 4.1. Else, Theorems 3.2 and 4.1 tell us that

$$\frac{M_{3r}(u_\lambda)}{M_{2r}(u_\lambda)}\leqslant C_ge^{C_2R\sqrt{\lambda}}\left(\frac{M_{8R}(u_\lambda)}{M_{3R}(u_\lambda)}\right)^2\leqslant C_g(d_M)e^{2C_2d_M\sqrt{\lambda}}.\qquad\square$$

4.3. *Outline of the proof of Yau's conjecture for real analytic manifolds*

In this section, we illustrate the importance of Theorem 4.2. We recall the following result.

CONJECTURE 4.3 [16]. Let M be a C^∞ closed compact Riemannian manifold. Let u_λ be a λ-eigenfunction on M. Then

$$C_1\sqrt{\lambda}\leqslant \mathrm{Vol}_{n-1}(\{u_\lambda=0\})\leqslant C_2\sqrt{\lambda},$$

where C_1 and C_2 are constants independent of λ.

The conjecture was proved in the case of real analytic Riemannian metrics in [6]. A major ingredient in the proof of both bounds was Theorem 4.2. We outline here the main ideas involved.

Lower bound. Let $B \subset M$ be a ball of radius $r = C/\sqrt{\lambda}$ such that u_λ vanishes at the centre of B. One can cover, say, $\frac{1}{2}$ of the volume of M by a disjoint collection $\mathcal{B}$ of such balls [5]. One observes that if the growth of u_λ in a ball B is smaller than, say, 20, then one can control from below the size of the nodal set in B. This can be seen for harmonic functions in the unit ball using the mean value principle and the isoperimetric inequality, and can be adapted to eigenfunctions on balls of radius $C/\sqrt{\lambda}$.

The main claim is that on at least, say, 10% of the balls in the collection $\mathcal{B}$, the growth is bounded by 20. We sketch the main ideas involved in the proof of this claim.

(1) Let us assume $M = \{|x| < 30\} \subset \mathbb{R}^n$. One can continue the function u_λ to a holomorphic function F on $M \times M \subset \mathbb{C}^n$, with $F|_{M\times\{0\}} = u_\lambda$. We let $Q \subset M \times \{0\}$ be a Euclidean real cube. A key fact is that due to Theorem 4.2, the growth of F in $M \times M$ is controlled by $\sqrt{\lambda}$.

(2) We subdivide Q to small subcubes Q_ν of side $1/\sqrt{\lambda}$. In order to bound the growth of u_λ in a cube Q_ν by a constant independent of λ, it is enough to say that $u_\lambda(x)$ is close to the average of u on Q_ν for most of the points $x \in Q_\nu$.

(3) The last property behaves well under averaging. Therefore, it can be reduced to a one-dimensional problem:

$$Q = [-1, 1] \subset \mathbb{R}, \quad B = \{|z| < 2\} \subset \mathbb{C},$$

where F is a holomorphic function defined on B, $F|_Q$ is real, and the growth of F in B is bounded by $\sqrt{\lambda}$.

(4) For simplicity, let us assume that F is a polynomial (of one complex variable) of degree $\sqrt{\lambda}$. We subdivide Q into subintervals Q_ν of size $1/\sqrt{\lambda}$. One has to show that $F|_{Q_\nu}$ is close to its average on Q_ν for at least 10% of these subintervals. Use the Hilbert transform to attack the one-dimensional problem.

Upper bound. As before, we assume $M = \{|x| < 30\} \subset \mathbb{R}^n$. The size of the nodal set is estimated from above by an integral geometry argument. One needs to estimate from above the number of zeros of u_λ on intervals $J \subset M$. As in the lower bound case, we let F be the holomorphic continuation of u_λ to $M \times M \subset \mathbb{C}^n$. Given an interval $J \subset M$, one can consider its complexification in $M \times M$. This is a strip of real two dimensions, $\hat{J} \subset \mathbb{C}^n$, and $F|_{\hat{J}}$ is holomorphic. One uses Jensen's formula on the function $F|_{\hat{J}}$ in order to estimate from above the number of zeros of F on J. In order to use Jensen's formula one must have a bound on the growth of $F|_{\hat{J}}$. This is where Theorem 4.2 comes in.

5. *Proof of Theorem* 2.2

5.1. *Preliminary geometric estimates*

Let N be a Riemannian manifold of dimension n. Fix a point p, and let $r(x) = \operatorname{dist}(x, p)$. Let

$$\gamma_K = \Delta r - (n-1)\cot_K r;$$

γ_K is controlled by the curvature of N.

LEMMA 5.1. *If* $\kappa \leqslant \mathrm{Sec}_N \leqslant K$, *then* $0 \leqslant \gamma_K \leqslant (n-1)(\cot_\kappa r - \cot_K r)$.

Proof. Both parts directly follow from the Hessian comparison theorem [4, 15]. □

Lemma 5.2. *Suppose $\kappa \leqslant \mathrm{Sec}_N \leqslant K$. Then, we have*

$$\gamma_{K,r} \geqslant -(n-1)(K-\kappa).$$

Proof. We know [**14**, Chapter 9.1]

$$\gamma_{K,r} = (\Delta r)_r + \frac{n-1}{(\sin_K r)^2} = -\mathrm{Ric}(\partial_r, \partial_r) - \|\mathrm{Hess}(r)\|^2 + \frac{n-1}{(\sin_K r)^2}.$$

By the Hessian comparison theorem [**15**],

$$(\cot_K r)\|X\|^2 \leqslant \mathrm{Hess}(r)(X,X) \leqslant (\cot_\kappa r)\|X\|^2. \tag{5.1}$$

Hence,

$$|\mathrm{Hess}(r)(X,X)|^2 \leqslant (\cot_\kappa r)^2\|X\|^4.$$

We can choose an orthonormal basis $(\partial_r, e_1, \ldots, e_{n-1})$ in which $\mathrm{Hess}(r)$ is diagonalized. Then we see

$$\|\mathrm{Hess}(r)\|^2 = \sum |\mathrm{Hess}(r)(e_i,e_i)|^2 \leqslant (n-1)(\cot_\kappa r)^2.$$

Consequently,

$$\begin{aligned}\gamma_{K,r} &\geqslant -(n-1)K - (n-1)(\cot_\kappa r)^2 + \frac{n-1}{(\sin_K r)^2} = (n-1)(\cot_K^2 r - \cot_\kappa^2 r)\\ &\geqslant -(n-1)(K-\kappa),\end{aligned}$$

where the last inequality follows from Lemma 5.3 (iii), (iv). □

Lemma 5.3.

(i) $-\frac{1}{3} \leqslant (\sqrt{x}\cot\sqrt{x})' \leqslant 0$ *for all* $0 \leqslant x < (\pi/2)^2$;
(ii) $0 \leqslant (\sqrt{x}\coth\sqrt{x})' \leqslant \frac{1}{3}$ *for all* $x \geqslant 0$;
(iii) $-1 \leqslant (x\cot^2\sqrt{x})' \leqslant 0$ *for all* $0 \leqslant x < (\pi/2)^2$;
(iv) $0 \leqslant (x\coth^2\sqrt{x})' \leqslant 1$ *for all* $x \geqslant 0$.

Proof. We prove the right inequality in (ii): since $y\coth y \geqslant 1$, we have $(3y + 2y\sinh^2 y)' \geqslant 3(\cosh y \sinh y)'$. Integrating, we conclude

$$3y + 2y\sinh^2 y \geqslant 3\cosh y \sinh y.$$

Equivalently, $(y\coth y)' \leqslant 2y/3$. Hence, $(\sqrt{x}\coth\sqrt{x})' \leqslant \frac{1}{3}$.

We prove the left inequality in (iii):

$$(x\cot^2\sqrt{x})' = \cot^2\sqrt{x} - \frac{\sqrt{x}\cot\sqrt{x}}{\sin^2\sqrt{x}}. \tag{5.2}$$

Observe that for $0 \leqslant y < \pi/2$,

$$y\cot y \leqslant 1. \tag{5.3}$$

From (5.2) and (5.3), it follows that

$$(x\cot^2\sqrt{x})' \geqslant \cot^2\sqrt{x} - \frac{1}{\sin^2\sqrt{x}} = -1.$$

The proofs of all the other inequalities in the lemma are omitted. □

5.2. *Choice of coordinates and notation*

We take geodesic polar coordinates centred at $p \in N$. Fix any $K \in \mathbb{R}$. The metric can be written as

$$g = dr^2 + (\sin_K r)^2 (a_K)_{ij}\, d\theta^i\, d\theta^j,$$

where θ^i are coordinates on the standard unit sphere $S^{n-1} \subset \mathbb{R}^n$.

We denote the determinant of the matrix $(a_K)_{ij}$ by a_K. The Laplacian on N can be written as

$$(\Delta f)(r,\theta) = f_{rr}(r,\theta) + ((n-1)\cot_K r + \gamma_K) f_r(r,\theta) + \frac{1}{(\sin_K r)^2} \left(\Delta_S f(r,\cdot)\right)(\theta),$$

where Δ_S is the following operator acting on functions g defined on S^{n-1}:

$$(\Delta_S g)(\theta) := \frac{1}{\sqrt{a_K}} \frac{\partial}{\partial\theta^i} \left(a_K^{ij} \sqrt{a_K} \frac{\partial g}{\partial \theta^j} \right).$$

We emphasize that the definition of Δ_S depends on our choice of K. With these definitions, we also have

$$\gamma_K = \frac{(\sqrt{a_K})_r}{\sqrt{a_K}}.$$

5.3. *Proof of Theorem 2.2(i)*

We observe

$$q(r) = \int u^2 (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta,$$

where the integration is understood to be performed over the parameter space $[0,\pi]^{n-2} \times [0, 2\pi]$ for S^{n-1} in $\mathbb{R}^{n-1}$. A straightforward computation shows the following.

LEMMA 5.4.

$$\begin{aligned} q'(r) = &\int 2uu_r (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta + \int u^2 \gamma_K (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta \\ &+ (n-1)(\cot_K r) \int u^2 (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta. \end{aligned}$$

LEMMA 5.5.

$$\int 2uu_r (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta \geqslant 0.$$

Proof. By Green's formula and the harmonicity of u

$$\begin{aligned} \int 2uu_r (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta &= \int_{\partial B(p,r)} \frac{\partial (u^2)}{\partial \hat{n}}\, dA_r \\ &= \int_{B(p,r)} \Delta(u^2)\, d\operatorname{Vol} = \int_{B(p,r)} 2|\nabla u|^2\, d\operatorname{Vol}. \end{aligned}$$ □

Proof of Theorem 2.2(i). Part (i) of the theorem follows directly from Lemmas 5.4, 5.5 and 5.1. □

5.4. *Proof of Theorem 2.2* (ii)

Let $w = (\sin_K r)^l u$, where $l = (n-2)/2$, then w satisfies the equation

$$w_{rr} + (\cot_K r + \gamma_K) w_r + l(l+1)Kw - \frac{l^2 w}{(\sin_K r)^2} + \frac{\Delta_S w}{(\sin_K r)^2} = 0. \tag{5.4}$$

Let

$$Q(r) = \int w(r,\theta)^2 \sqrt{a_K}\, d\theta = \frac{q(r)}{\sin_K(r)}. \tag{5.5}$$

Let us also set

$$\nabla_S w := (\sin_K r)\left(\nabla w - w_r \frac{\partial}{\partial r}\right) = \frac{1}{\sin_K r} a_K^{ij} \frac{\partial w}{\partial \theta^i} \frac{\partial}{\partial \theta^j};$$

note that ∇_S is defined in this way in order to have Green's formula:

$$\int f(\theta)(\Delta_S g)(\theta)\sqrt{a_K}\, d\theta = -\int \langle \nabla_S f, \nabla_S g\rangle \sqrt{a_K}\, d\theta. \tag{5.6}$$

Note also that $\langle \nabla_S w, \partial_r\rangle = 0$.

Lemma 5.6.

(i) $Q'(r) = \int 2w(w_r + \gamma_K w/2)\sqrt{a_K}\, d\theta$;
(ii) $Q'(r) \geqslant (n-2)(\cot_K r)Q(r) \geqslant 0$.

Proof. Part (i) is a direct calculation. Part (ii) is just another formulation of Theorem 2.2(i). □

A second direct calculation using equation (5.4) and formula (5.6) gives the following lemma.

Lemma 5.7.

$$\begin{aligned} Q''(r) + (\cot_K r)Q'(r) = {} & 2\int \left(w_r + \frac{\gamma_K}{2} w\right)^2 \sqrt{a_K}\, d\theta + \frac{2}{(\sin_K r)^2}\int |\nabla_S w|^2\, d\theta \\ & + \frac{2l^2}{(\sin_K r)^2}\int w^2 \sqrt{a_K}\, d\theta - 2l(l+1)K\int w^2\sqrt{a_K}\, d\theta \\ & + \int w^2 \left(\gamma_{K,r} + \gamma_K \cot_K r + \frac{\gamma_K^2}{2}\right)\sqrt{a_K}\, d\theta. \end{aligned}$$

Lemma 5.8.

$$\begin{aligned} Q''(r) + (\cot_K r)Q'(r) \geqslant {} & 2\int \left(w_r + \frac{\gamma_K}{2} w\right)^2 \sqrt{a_K}\, d\theta \\ & + \frac{2}{(\sin_K r)^2}\int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + \frac{2l^2}{(\sin_K r)^2} Q \\ & - 2l(l+1)KQ - (n-1)(K-\kappa)Q. \end{aligned}$$

Proof. This estimate is due to Lemma 5.7 and the estimates on γ_K and $\gamma_{K,r}$ in Lemmas 5.1 and 5.2, respectively. □

Immediately, we obtain the following result.

LEMMA 5.9.

$$\begin{aligned} Q''(r) + (\cot_K r)Q'(r) - \frac{Q'(r)^2}{Q(r)} \geqslant\ & 2\int \left(w_r + \frac{\gamma_K}{2} w\right)^2 \sqrt{a_K}\, d\theta \\ & + \frac{2}{(\sin_K r)^2} \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta \\ & + \frac{2l^2}{(\sin_K r)^2} Q - 2l(l+1)KQ - (n-1)(K-\kappa)Q \\ & - \frac{4\left(\int w(w_r + \gamma_K w/2)\sqrt{a_K}\, d\theta\right)^2}{\int w^2 \sqrt{a_K}\ d\theta}. \end{aligned}$$

LEMMA 5.10.

$$\begin{aligned} & Q''(r) + (\cot_K r)Q'(r) - \frac{Q'(r)^2}{Q(r)} + (n-1)(\cot_\kappa r - \cot_K r)Q'(r) \\ & \geqslant \frac{\varphi(r)}{(\sin_K r)^2} + \frac{2l^2}{(\sin_K r)^2} Q - 2l(l+1)KQ - (n-1)(K-\kappa)Q, \end{aligned}$$

where

$$\varphi(r) = -2(\sin_K r)^2 \int w_r^2 \sqrt{a_K}\, d\theta + 2\int |\nabla_S w|^2 \sqrt{a_K}\, d\theta.$$

Proof.

$$\begin{aligned} & Q''(r) + (\cot_K r)Q'(r) - \frac{Q'(r)^2}{Q(r)} \\ & \geqslant 2\int \left(w_r + \frac{\gamma_K}{2} w\right)^2 \sqrt{a_K}\, d\theta + \frac{2}{(\sin_K r)^2} \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta \\ & \quad + \frac{2l^2}{(\sin_K r)^2} Q - 2l(l+1)KQ - (n-1)(K-\kappa)Q \\ & \quad - \frac{2\left(\int w(w_r + \gamma_K w/2)\sqrt{a_K}\, d\theta\right)^2}{\int w^2 \sqrt{a_K}\ d\theta} - \frac{2\left(\int w(w_r + \gamma_K w/2)\sqrt{a_K}\, d\theta\right)^2}{\int w^2 \sqrt{a_K}\ d\theta} \\ & \geqslant \frac{2}{(\sin_K r)^2} \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + \frac{2l^2}{(\sin_K r)^2} Q - 2l(l+1)KQ \\ & \quad - (n-1)(K-\kappa)Q - \frac{2\left(\int w w_r \sqrt{a_K}\, d\theta + \int \gamma_K w^2/2 \sqrt{a_K}\, d\theta\right)^2}{\int w^2 \sqrt{a_K}\ d\theta} \\ & \geqslant \frac{\varphi(r)}{(\sin_K r)^2} + \frac{2l^2}{(\sin_K r)^2} Q - \frac{\int \gamma_K w^2 \sqrt{a_K}\, d\theta}{\int w^2 \sqrt{a_K}\, d\theta} Q' + \frac{(\int \gamma_K w^2 \sqrt{a_K}\, d\theta)^2}{2\int w^2 \sqrt{a_K}\, d\theta} \\ & \quad - 2l(l+1)KQ - (n-1)(K-\kappa)Q \end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{\varphi(r)}{(\sin_K r)^2} + \frac{2l^2}{(\sin_K r)^2} Q \\
&\quad - (n-1)(\cot_\kappa r - \cot_K r)Q' - 2l(l+1)KQ - (n-1)(K-\kappa)Q.
\end{aligned}
$$

The first inequality is just a rewriting of Lemma 5.9. In the second inequality, we applied the Cauchy–Schwarz inequality to the last term. In the third inequality, we unfolded the parentheses in the last term and applied the Cauchy–Schwarz inequality to the term $\int w w_r \sqrt{a_K}\, d\theta$. In the last inequality, we used the fact that $Q' \geqslant 0$ (Lemma 5.6) and the estimates on γ_K in Lemma 5.1. □

It remains to control the function φ in terms of Q and Q'. We would first like to calculate the derivative of φ. To that end, we recall the definition and some of the properties of the Hessian as a bilinear form:

$$
\mathrm{Hess} f(X,Y) := XYf - (\nabla_X Y)f = \langle Y, \nabla_X \mathrm{grad} f\rangle = \langle X, \nabla_Y \mathrm{grad} f\rangle.
$$

In a geodesic ball centred at p, we have a radial field $\mathrm{grad}\, r = \partial_r$, tangent to the geodesics emanating from p. Since ∂_r is tangent to a geodesic, we have $\nabla_{\partial_r}\partial_r = 0$. As a consequence, $(\mathrm{Hess}\, r)(\partial_r, Y) = 0$ for all vectors Y. When computing the derivative of φ, it is convenient to have the following formula.

Lemma 5.11.

$$
(|\nabla_S f|^2)_r = 2\langle \nabla_S f, \nabla_S f_r\rangle - 2\,\mathrm{Hess}(r)(\nabla_S f, \nabla_S f) + 2(\cot_K r)|\nabla_S f|^2.
$$

Proof.

$$
\begin{aligned}
2\,\mathrm{Hess}(r)(\nabla_S f, \nabla_S f) &= 2(\sin_K r)^2\, \mathrm{Hess}(r)(\nabla f, \nabla f) \\
&= 2(\sin_K r)^2 \langle \nabla f, \nabla_{\nabla f}\partial_r\rangle = 2(\sin_K r)^2\langle \nabla f, \nabla_{\partial_r}\nabla f + [\nabla f, \partial_r]\rangle \\
&= (\sin_K r)^2(|\nabla f|^2)_r + 2(\sin_K r)^2[\nabla f, \partial_r] f \\
&= -(\sin_K r)^2(|\nabla f|^2)_r + 2(\sin_K r)^2\langle \nabla f, \nabla f_r\rangle \\
&= -(\sin_K r)^2\left(f_r^2 + (\sin_K r)^{-2}|\nabla_S f|^2\right)_r + 2(\sin_K r)^2 f_r f_{rr} \\
&\quad + 2\langle \nabla_S f, \nabla_S f_r\rangle \\
&= -(|\nabla_S f|^2)_r + 2(\cot_K r)|\nabla_S f|^2 + 2\langle \nabla_S f, \nabla_S f_r\rangle.
\end{aligned}
$$

□

Using the formula in Lemma 5.11, we can readily compute the derivative of $\varphi(r)$ (defined in Lemma 5.10).

Lemma 5.12.

$$
\begin{aligned}
\varphi'(r) = &-4\int \mathrm{Hess}(r)(\nabla_S w, \nabla_S w)\sqrt{a_K}\, d\theta \\
&+ 4(\cot_K r)\int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + 2l(l+1)K(\sin_K r)^2 Q' - 2l^2 Q' \\
&+ 2(\sin_K r)^2 \int |\nabla w|^2 \gamma_K \sqrt{a_K}\, d\theta \\
&+ 2l^2 \int w^2 \gamma_K \sqrt{a_K}\, d\theta - 2l(l+1)K \sin_K^2 r \int w^2 \gamma_K \sqrt{a_K}\, d\theta.
\end{aligned}
$$

LEMMA 5.13.

$$\varphi'(r) \geqslant -4(\cot_\kappa r - \cot_K r)\int |\nabla_S w|^2 \sqrt{a_K}\, d\theta + 2l(l+1)K(\sin_K r)^2 Q' - 2l^2 Q' - 2l(l+1)(n-1)(\cot_\kappa r - \cot_K r)K^+(\sin_K r)^2 Q.$$

Proof. This is due to inequality (5.1) and Lemma 5.12. □

In Lemma 5.17, we integrate the inequality in Lemma 5.13. We need a few lemmas before that.

LEMMA 5.14.

$$\left(1 - \frac{2}{n}\right)(\sin_K r)^2 Q(r) \leqslant \int_0^r (\sin_K \rho)^2 Q'(\rho)\, d\rho \leqslant (\sin_K r)^2 Q(r).$$

Proof. The right-hand side follows from the fact that $\sin_K \rho$ is monotonically increasing in ρ, and $Q' \geqslant 0$. By differentiating the left-hand side, we see that it is enough to prove

$$\left(1 - \frac{2}{n}\right)(\sin_K r)^2 (2(\cot_K r)Q(r) + Q'(r)) \leqslant (\sin_K r)^2 Q'(r). \tag{5.7}$$

Inequality (5.7) is equivalent to Lemma 5.6(ii). □

LEMMA 5.15.

$$\int_0^r \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta\, d\rho \leqslant \frac{(\sin_K r)^2}{2}\left(Q'(r) - (n-2)(\cot_K r)Q\right).$$

Proof.

$$\begin{aligned}
\int_0^r \int |\nabla_S w|^2 \sqrt{a_K}\, d\theta\, d\rho &= \int_0^r \int |\nabla_S u|^2 (\sin_K \rho)^{n-2} \sqrt{a_K}\, d\theta\, d\rho \\
&\leqslant \int_0^r \int |\nabla u|^2 (\sin_K \rho)^n \sqrt{a_K}\, d\theta\, d\rho \\
&\leqslant \sin_K r \int_0^r \int |\nabla u|^2 (\sin_K \rho)^{n-1} \sqrt{a_K}\, d\theta\, d\rho \\
&= \sin_K r \int_{B(p,r)} |\nabla u|^2\, d\,\mathrm{Vol} = \sin_K r \int u u_r (\sin_K r)^{n-1} \sqrt{a_K}\, d\theta \\
&= (\sin_K r)^2 \int w w_r \sqrt{a_K}\, d\theta - l \cot_K r (\sin_K r)^2 \int w^2 \sqrt{a_K}\, d\theta \\
&= (\sin_K r)^2 \int w(w_r + \gamma_K w/2)\sqrt{a_K}\, d\theta \\
&\quad - l(\cot_K r)(\sin_K r)^2 \int w^2 \sqrt{a_K}\, d\theta - (\sin_K r)^2 \int w^2 \gamma_K/2 \sqrt{a_K}\, d\theta \\
&\leqslant \frac{(\sin_K r)^2}{2}(Q'(r) - (n-2)(\cot_K r)Q).
\end{aligned}$$

□

LEMMA 5.16.

$$\int_0^r (\sin_K \rho)^2 Q(\rho)\, d\rho \leqslant r(\sin_K r)^2 Q(r).$$

Proof. The functions $\sin_K \rho$ and $Q(\rho)$ are monotonically increasing in ρ. $\square$

LEMMA 5.17.

$$\begin{aligned}\frac{\varphi(r)}{(\sin_K r)^2} \geqslant{}& -2(\cot_\kappa r - \cot_K r)\left(Q' - (n-2)(\cot_K r)Q\right)\\ &+ \frac{n(n-2)}{2}KQ - (n-2)K^+Q - \frac{(n-2)^2 Q}{2(\sin_K r)^2}\\ &- \frac{n}{2}(n-1)(n-2)(\cot_\kappa r - \cot_K r)rK^+Q.\end{aligned}$$

Proof. Observe that the functions $\cot_\kappa r - \cot_K r$ and $\sin_K r$ are both monotonically increasing. Hence, integrating Lemma 5.13, while applying Lemmas 5.14–5.16, we obtain

$$\begin{aligned}\varphi(r) \geqslant{}& -4(\cot_\kappa r - \cot_K r)\int_0^r\!\!\int |\nabla_S w|^2 \sqrt{a_K}\, d\theta\, d\rho\\ &+ 2l(l+1)K\int_0^r (\sin_K \rho)^2 Q'(\rho)\, d\rho - 2l^2 Q\\ &- 2l(\cot_\kappa r - \cot_K r)K^+ \int_0^r (\sin_K \rho)^2 Q(\rho)\, d\rho\\ \geqslant{}& -2(\sin_K r)^2(\cot_\kappa r - \cot_K r)\left(Q' - (n-2)(\cot_\kappa r - \cot_K r)(\cot_K r)Q\right)\\ &+ 2l(l+1)K(\sin_K r)^2 Q(r) - 2l(l+1)K^+(2/n)(\sin_K r)^2 Q - 2l^2 Q\\ &- (n-2)(\cot_\kappa r - \cot_K r)rK^+(\sin_K r)^2 Q(r).\end{aligned}$$ $\square$

Proof of Theorem 2.2(ii). From Lemmas 5.10 and 5.17, we obtain

$$\begin{aligned}&Q''(r) + (\cot_K r)Q'(r) - \frac{Q'(r)^2}{Q(r)} + (n-1)(\cot_\kappa r - \cot_K r)Q'(r)\\ &\geqslant -2(\cot_\kappa r - \cot_K r)Q'(r) + 2(n-2)(\cot_\kappa r - \cot_K r)(\cot_K r)Q\\ &\quad + \frac{n(n-2)}{2}KQ(r) - (n-2)K^+Q - \frac{(n-2)^2}{2(\sin_K r)^2}Q\\ &\quad - (n-2)(\cot_\kappa r - \cot_K r)rK^+Q - \frac{n(n-2)}{2}KQ\\ &\quad + \frac{(n-2)^2}{2(\sin_K r)^2}Q - (n-1)(K-\kappa)Q\\ &= -2(\cot_\kappa r - \cot_K r)Q'(r) + 2(n-2)(\cot_\kappa r - \cot_K r)(\cot_K r)Q\\ &\quad - (n-2)K^+Q - (n-2)(\cot_\kappa r - \cot_K r)rK^+Q - (n-1)(K-\kappa)Q.\end{aligned} \tag{5.8}$$

We obtain

$$\begin{aligned}
&(\log Q)''(r) + (\cot_K r)(\log Q)'(r) + (n+1)(\cot_\kappa r - \cot_K r)(\log Q)'(r)\\
&\geqslant -(n-1)(K-\kappa) - (n-2)K^+ - (n-2)(K-\kappa)\frac{r^2\,K^+}{3}\\
&\geqslant -(2n-3)(K-\kappa) - (n-2)K^+, \qquad (5.9)
\end{aligned}$$

where we applied Lemma 5.3(i),(ii). Recall $q(r) = Q(r)(\sin_K r)$. A direct computation shows

$$\begin{aligned}
&(\log \sin_K r)'' + (\cot_K r)(\log \sin_K r)' + (n+1)(\cot_\kappa r - \cot_K r)(\log \sin_K r)'\\
&= -K + (n+1)\cot_K r(\cot_\kappa r - \cot_K r) \geqslant -K. \qquad (5.10)
\end{aligned}$$

Finally, summing (5.9) and (5.10) gives the statement in the theorem. □

5.5. *Proof of Corollary* 2.3

Proof of Corollary 2.3. From Theorem 2.2,

$$q''(r) + (\cot_K r)q' + (n+1)(\cot_{-K} r - \cot_K r)q'(r) \geqslant -(5n-7)Kq. \qquad (5.11)$$

From Lemma 5.3 and from the fact that $q' \geqslant 0$ (Theorem 2.2(i)) we know

$$(n+1)(\cot_{-K} r - \cot_K r)q' \leqslant 2(n+1)rKq'/3. \qquad (5.12)$$

From Theorem 2.2(i), we know

$$-(5n-7)Kq \geqslant -5Kq'(r)/\cot_K r. \qquad (5.13)$$

It is easy to check $1/\cot_K r \leqslant 2r$ for $r \leqslant \pi/(3\sqrt{K})$.

Hence, from inequalities (5.11)–(5.13) we obtain

$$q''(r) + \frac{1+8nr^2K}{r}q' - \frac{q'(r)^2}{q(r)} \geqslant 0 \qquad (5.14)$$

for $r\sqrt{K} < \pi/3$. If we define $l(t) = q(e^t)$, then (5.14) is equivalent to

$$l''(t) + 8nKe^{2t}l'(t) \geqslant 0 \qquad (5.15)$$

for $t < -(\log K)/2 + \log(\pi/3)$. We will now integrate inequality (5.15).

Inequality (5.15) can be rewritten as $(e^{4nKe^{2t}}l'(t))' \geqslant 0$, from which we see that for $s_2 < s_1$,

$$l'(s_2) \leqslant e^{4nK(e^{2s_1}-e^{2s_2})}l'(s_1) \leqslant e^{4nKe^{2s_1}}l'(s_1), \qquad (5.16)$$

where the last inequality is true since $l'(s) \geqslant 0$ from Theorem 2.2(i). Hence, for $t_2 < t_1$ such that $16nKe^{2t_1} < 1$, and $0 \leqslant h \leqslant \log 2$,

$$\begin{aligned}
l(t_2+h) - l(t_2) &= \int_0^h l'(t_2+s)\,ds \leqslant \int_0^h e^{4nKe^{2t_1+2s}}l'(t_1+s)\,ds\\
&\leqslant e^{4nKe^{2t_1+2h}}(l(t_1+h) - l(t_1))\\
&\leqslant (1+32nKe^{2t_1})(l(t_1+h) - l(t_1)).
\end{aligned}$$

The last inequality follows from $e^x \leqslant 1+2x$ for $0 \leqslant x \leqslant 1$.

Going back from the variable t to the variable r, we obtain the stated corollary. □

6. *The case of constant curvature manifolds*

We give a new proof of Theorem 2.1 and a second proof of Theorem 2.2 in the case of constant non-zero curvature in two dimensions.

6.1. *Zero curvature*

Let $u_l(r,\theta) = r^l \cos(l\theta)$ and $v_l = r^l \sin(l\theta)$; then $q_{u_l}(r) = q_{v_l}(r) = \pi r^{2l+1}$. It is obvious that $\log q_l$ is a convex function of $\log r$.

Now, any harmonic function can be written as

$$u = a_0 + \sum_{l=1}^{\infty} a_l u_l(r,\theta) + b_l v_l(r,\theta).$$

The functions $u_l(r,\cdot)$ and $v_l(r,\cdot)$ are pairwise orthogonal as functions on the unit circle for all fixed r. For any two orthogonal functions f and g on the unit circle for all fixed r, we have $q_{f+g}(r) = q_f(r) + q_g(r)$. We also know that the sum of log-convex functions is log-convex and the pointwise limit of log-convex functions is log-convex. These considerations give a short new proof of Theorem 2.1.

Remark. A similar argument also carries in dimensions greater than or equal to three.

6.2. *Positive curvature, two dimensions*

The metric on the two-dimensional sphere of constant curvature $K > 0$ is given by

$$ds^2 = dr^2 + (\sin_K r)^2\, d\theta^2.$$

Here, $0 < r < \pi/\sqrt{K}$, and $0 \leqslant \theta \leqslant 2\pi$. Hence,

$$q_u^K(r) = \int_0^{2\pi} u(r,\theta)^2 (\sin_K r)\, d\theta.$$

We also define $q_f^0(r) = \int_0^{2\pi} f(r,\theta)^2 r\, d\theta$ for functions defined on $\mathbb{R}^2$.

Let $f(r,\theta)$ be defined on $\mathbb{R}^2$ by $u(r,\theta) = f(\tan(r\sqrt{K}/2),\theta)$; then f is related to u by a stereographic projection. Since harmonic functions are preserved under conformal transformations in two dimensions, $f(r,\theta)$ is harmonic if and only if $u(r,\theta)$ is harmonic. We also note the relation

$$q_u^K(r) = \frac{q_f^0(\tan(r\sqrt{K}/2))}{\tan(r\sqrt{K}/2)} \sin_K r.$$

Suppose now f is harmonic. Then, from the fact that $\log q_f^0$ is a convex function of $\log r$, we obtain the following.

Theorem 6.1. *If $K > 0$, then*

$$(\log q_u^K)''(r) + (\cot_K r)(\log q_u^K)'(r) \geqslant -K.$$

6.3. *Negative curvature*

In the spherical example one can replace all trigonometric functions by the corresponding hyperbolic functions and obtain the following result.

THEOREM 6.2. *If $K < 0$, then*

$$(\log q_u^K)''(r) + (\cot_K r)(\log q_u^K)'(r) \geqslant -K > 0.$$

7. *Discussion*

We raise several questions which we find interesting to pursue.

7.1. *Beyond the injectivity radius*

It would be interesting to understand whether Theorem 2.2 remains true beyond the injectivity radius as long as $r\sqrt{K^+} < \pi/2$ in the spirit of Bishop–Gromov's volume comparison theorem [**8**].

7.2. *Proof by an orthogonal basis of functions*

In a manifold of constant curvature $K \neq 0$, of dimension greater than or equal to three, we would like to have a simple proof, inspired by the proof presented in Section 6 for the case $K = 0$. This would also shed light on the sharpness of Theorem 2.2 in dimensions $n \geqslant 3$.

7.3. *Ricci curvature*

Can one of the bound assumptions on the sectional curvature in Theorem 2.2 be relaxed to a bound on the Ricci curvature?

7.4. *Eigenfunctions on negatively curved manifolds*

Can we replace the extension procedure described in Section 3 by a procedure which will give us more information on the growth of eigenfunctions on negatively curved manifolds?

7.5. *A comparison theorem for positive harmonic functions*

Let $f(\theta)$ be a 2π-periodic non-negative function. Let u be a solution of the Dirichlet problem in the unit disk: $\Delta u = 0$ with $u(1, \theta) = f(\theta)$. Now, suppose we consider the unit geodesic disk in a Riemannian manifold with non-positive variable curvature, and solve the Dirichlet problem there. We obtain a solution $v(r, \theta)$. Can we compare the values of u to the values of v? Or equivalently, can we compare the Poisson kernels involved?

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