Preliminaries

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Let (M, g) be a smooth Riemannian manifold.

For $\lambda > 0$, we consider Laplace eigenfunctions, u_λ, satisfying

$$\Delta_M u_\lambda + \lambda u_\lambda = 0$$

As usual we will consider the harmonic function on $M \times [-1,1]$ defined by

$$h(x,t) = e^{\lambda^{1/2} t} u_\lambda(x)$$

i.e. $\Delta_M h + \partial_{tt} h = 0$

In local coordinates h satisfies the divergence form equation

$$\Delta_{M\times\mathbb{R}} h = \frac{1}{\sqrt{g}} \operatorname{div}\left(g^{1/2} (g^{ij}) \nabla h\right) = 0.$$

where (g^{ij}) is the metric on $M \times \mathbb{R}$.

Since h satisfies the uniformly elliptic equation, h satisfies many nice principles/estimates:

1.) Maximum Principle

2.) Gradient Estimates

$\exists C > 0$ s.t. for geodesic balls $B(x,r) \subset M \times [-1,1]$,

$$|\nabla h(x)| \leq \frac{C}{r} \sup_{B(x,r)} |h|.$$

3.) Harnack Inequality

$\exists C > 0$ s.t. if $h > 0$ in $B(x,r)$ then for $y \in B(x, \tfrac{1}{2} r)$

$$\frac{1}{C} h(y) < h(x) < C h(y)$$

or alternatively

$$\sup_{B(x,r)} h \geq c \sup_{B(x,\frac{r}{2})} |h|.$$

Since $h = e^{\sqrt{\lambda} t} u$ satisfies these standard estimates, u inherits the estimates at length scale $\lambda^{-1/2}$

<u>Lemma</u>: (Lemma 2.1 Log. Mal)

For $r \lesssim \lambda^{-1/2}$, if $\Delta_M u = -\lambda u$

then

i.) $\sup_{B_r} |u| \leq 2 \sup_{\partial B_r} |u|$

ii.) $\sup_{B_{\frac{1}{2}r}} |\nabla u| \lesssim \sup_{\partial B_r} \frac{|u|}{r}$

iii.) if $u \geq 0$ on $\overline{B_r}$

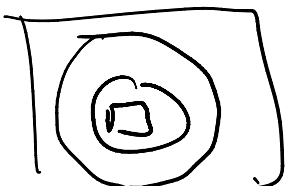
iii.) if $u \geq 0$ on B_r

$$\sup_{B_{\frac{2}{3}r}} |u| \lesssim \sup_{\partial B_r} u .$$

Furthermore, we have monotonicity properties of the doubling index

Again $\quad N(h, q) := \log \dfrac{\int_{2q} |h|^2}{\int_q |h|^2}$

where q is a cube.



Lemma:

i.) If a cube q is inscribed in a ball B (and thus $2q \supset \partial B$)

$$\sup_{\frac{4}{3}B} |h| \lesssim 2^{\frac{N(h,q)}{2}} \sup_{B} |h| .$$

ii.) (Monotonicity property)

$\exists\ A = A(d) \in \mathbb{Z}_+$, $C_0 = C_0(d) > 1$ and $\rho = \rho(M) > 0$ s.t. if

$$A q_1 \subset q \quad \text{then}$$

$$N(h, q_1) \leq C_0 N(h, q)$$

Pf: i.) follows by noting that

Pf: i.) Follows by noting that

$q \subset B \subset \frac{4}{3}B \subset lq$ and Jensen's inequality

ii.) Again, $q \subset B \subset \frac{4}{3}B \subset lq$ and use the monotonicity result of Garofalo and Lin.

$$\left(\begin{aligned} & N(u, q_1) \le C_0 N(u, q) \\ \Leftrightarrow \quad & \frac{\int_{lq_1} |u|^2}{\int_{q_1} |u|^2} \le \left(\frac{\int_{lq} |u|^2}{\int_q |u|^2} \right)^{C_0} \end{aligned} \right)$$

Garofalo and Lin consider the alternative definition of frequency given by

$$N_0(u, r) = \frac{r \int_{B_r} |\nabla u|^2}{\int_{\partial B_r} |u|^2}$$

which has the property that

$$r \frac{d}{dr} \left(\log \frac{\int_{\partial B_r} |u|^2}{r^{d-1}} \right) = 2 N_0(u, r).$$

and in our example

$$r \frac{d}{dr} \log \frac{\int_{\partial B_r} |u|^2}{r^{d-1}} = r \frac{d}{dr} \log n r^{2n}$$

$$\sim n.$$

In order to show almost monotonicity for solutions to the general elliptic equation $\operatorname{div}(A \nabla u) = 0$, they simply analyze the derivative of N_0 w.r.t. r.

$$\frac{d}{dr} N_0(u,r) = N_0(u,r) \cdot F(u,r)$$

For u satisfying $\Delta u = 0$, Algren showed that $F(u,r) \geq 0$

Garafalo and Lin show that there exists $C(d,A) > 0$ s.t. $F(u,r) \geq -C(d,A)$.