Monotonicity Properties of Variational Integrals, A_p Weights and Unique Continuation

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1. Introduction and statement of the results. Let Ω be a connected open subset of $\mathbf{R}^n$, $n \geq 3$, and let $A(x)$ be a symmetric $n \times n$ matrix-valued function on Ω on which we make the following assumptions:

(i) there exists a $\Gamma > 0$ such that for every $x, y \in \Omega$

$$|a_{ij}(x) - a_{ij}(y)| \leq \Gamma |x - y|, \quad i, j = 1, \dots, n; \tag{1.1}$$

(ii) there exists a $\lambda \in (0,1)$ such that for every $x \in \Omega$ and $\xi \in \mathbf{R}^n$

$$\lambda |\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \lambda^{-1} |\xi|^2. \tag{1.2}$$

We will consider solutions, u, to the equation

$$Lu = \operatorname{div}(A(x)\nabla u(x)) = 0 \qquad \text{in } \Omega. \tag{1.3}$$

By this we will mean that $u \in H^{1,2}_{\text{loc}}(\Omega)$ and that for every $\phi \in H^{1,2}_0(\Omega)$

$$\int_\Omega \langle A(x)\nabla u(x), \nabla \phi(x) \rangle dx = 0.$$

Under the assumptions (1.1) and (1.2) it is well known (see, e.g., [GT], Theorem 8.8) that every solution to (1.3) is in $H^{2,2}_{\text{loc}}(\Omega)$. Since all the results in this paper are of a local nature, we will assume henceforth that Ω strictly contains $\bar{B}_2$, the closure of the ball with radius 2 and center at the origin.

One of the main results of this paper can be stated as follows.

Theorem 1.1. *Let $u \in H^{1,2}_{\text{loc}}(\Omega)$ be a solution to (1.3). (i) If $u \not\equiv 0$ there exist a $p > 1$ and a constant $A > 0$ such that for any ball B_R, such that the concentric ball $B_{2R} \subset B_1$, we have*

$$\left(\frac{1}{|B_R|}\int_{B_R} |u| dx\right)\left(\frac{1}{|B_R|}\int_{B_R} |u|^{-1/(p-1)} dx\right)^{p-1} \leq A. \tag{1.4}$$

In (1.4) A and p depend on u, Γ, λ, n, but do not depend on B_R. (ii) If $u \not\equiv$ const. there exist a $q > 1$ and a constant $B > 0$ such that for any ball B_R as in (i)

$$\left(\frac{1}{|B_R|}\int_{B_R}|\nabla u|dx\right)\left(\frac{1}{|B_R|}\int_{B_R}|\nabla u|^{-1/(q-1)}dx\right)^{q-1}\le B. \tag{1.5}$$

B and q in (1.5) are independent of B_R, but they depend on u, Γ, λ and n.

The content of Theorem 1.1 can be summarized by saying that, under the stated conditions, $|u|$ and $|\nabla u|$ are, respectively, an A_p and an A_q *weight* of Muckenhoupt in B_1. A condition like (1.4), where $|u|$ is replaced by a nonnegative $w \in L^1_{\text{loc}}(\mathbf{R}^n)$, was first introduced by Muckenhoupt, [Muck], in order to characterize those measures wdx on $\mathbf{R}^n$ for which the Hardy-Littlewood maximal operator is continuous from $L^p(wdx)$ to itself. Our basic reference on A_p weights is the paper by Coifman and C. Fefferman, [CF]. Strongly related to (1.4) is the following so-called *reverse Hölder inequality*. Let $w \ge 0$ be an L^1_{loc} function in B_1. w is said to satisfy a reverse Hölder inequality in B_1 if there exist δ and C positive such that for every ball B_R, with $B_{2R} \subset B_1$,

$$\left(\frac{1}{|B_R|}\int_{B_R}w^{1+\delta}dx\right)^{1/(1+\delta)}\le C\left(\frac{1}{|B_R|}\int_{B_R}wdx\right). \tag{1.6}$$

In [CF] it is shown that if w satisfies (1.6) then it satisfies a condition like (1.4) for some $p > 1$, and that, conversely, (1.4) implies (1.6) for some $\delta > 0$.

In Section 3 below we prove that if $u \not\equiv 0$ is a solution to (1.3), then u^2 satisfies (1.6) for every $\delta > 0$. Moreover, if $u \not\equiv$ const. we prove that $|\nabla u|^s$ with $s = 2n/(n+2)$ satisfies a condition like (1.6) for a certain $\delta > 0$. From this, and the above-mentioned results in [CF], we obtain (1.4) and (1.5).

Another result in this paper is a theorem of strong unique continuation. Before stating it we need to recall the relevant definition. A function $u \in L^2_{\text{loc}}(\Omega)$ is said *to vanish of infinite order at $x_0 \in \Omega$* if for $R > 0$ sufficiently small

$$\int_{|x-x_0|<R}u^2dx = O(R^N), \qquad \text{for every } N \in \mathbf{N}. \tag{1.7}$$

Theorem 1.2. *Let $u \in H^{1,2}_{\text{loc}}(\Omega)$ be a solution to (1.3).*
(i) If u vanishes of infinite order at $x_0 \in \Omega$, then $u \equiv 0$ in Ω.
(ii) $|\nabla u|$ cannot vanish of infinite order at $x_0 \in \Omega$, unless $u \equiv$ const. in Ω.

We will prove Theorems 1.1 and 1.2 in Section 3 below. The proofs heavily rely on the establishment of the following

Theorem 1.3. (Doubling condition). *Let $u \in H^{1,2}_{\text{loc}}(\Omega)$ be a solution to (1.3). Then there exists a positive constant C, depending on u, Γ, λ and n, such that for any ball B_R, with $B_{2R} \subset B_1$, we have*

$$\int_{B_{2R}}u^2dx \le C\int_{B_R}u^2dx. \tag{1.8}$$

It is easy to see that C in (1.8) cannot be taken independent of u. In fact, if in $\mathbf{R}^2$ we consider the harmonic functions $u_k(r,\theta) = r^k\cos k\theta$, then the corresponding

C_k in (1.8) blows up like 2^k for balls B_R, B_{2R} centered at the origin.

Theorem 1.3 is the crucial part of the paper. Its main thrust consists in the fact that no sign assumption is made on u. It may be worthwhile to remark that for a nonnegative solution u, (1.8) is a simple consequence of Harnack's inequality, see, e.g., [M]. However, if u has arbitrary sign the situation is drastically different, as one has to control the zeros of u. In a sense that will be made precise in Section 3, this is exactly what (1.8) does.

At this point, to make our discussion clear without getting involved with the technical complications which arise in the general case, we make the simplifying assumption that $L = \Delta$, the Laplace operator in $\mathbf{R}^n$, and that u is a harmonic function in Ω, i.e., $\Delta u = 0$. For a ball B_r centered at the origin we consider the quantity

$$H(r) = \int_{\partial B_r} u^2 dH_{n-1}, \tag{1.9}$$

where dH_{n-1} denotes $(n-1)$-dimensional Hausdorff measure on ∂B_r. It turns out that $H(r)$ is related in a natural way, via the equation and the divergence theorem, to the Dirichlet integral

$$D(r) = \int_{B_r} |\nabla u|^2 dx, \tag{1.10}$$

of u over B_r. In fact, since $\Delta u = 0$ implies $\Delta(u^2) = 2|\nabla u|^2$, setting $u_\rho = \langle \nabla u, x/\rho \rangle$, $\rho = |x|$, we get from the divergence theorem

$$D(r) = \frac{1}{2}\int_{B_r} \Delta(u^2)dx = \int_{\partial B_r} uu_\rho dH_{n-1}. \tag{1.11}$$

On the other hand, a computation yields

$$\frac{dH}{dr} = \frac{n-1}{r}H(r) + 2\int_{\partial B_r} uu_\rho dH_{n-1}, \tag{1.12}$$

thus from (1.12) and (1.11) we obtain

$$\frac{d}{dr}\left(\log \frac{H(r)}{r^{n-1}}\right) = 2\frac{D(r)}{H(r)}. \tag{1.13}$$

The crucial claim is that, if we set $N(r) = rD(r)/H(r)$, then

(1.14) *$N(r)$ is a nondecreasing function of r.*

(1.14) was first observed by F. J. Almgren, Jr., [Al], who called $N(r)$ the *frequency* of the harmonic function u. The reason for the name is that if in $\mathbf{R}^2$ we consider $u_k(r,\theta) = a_k r^k \sin k\theta$, then $N(r) \equiv k$. Using (1.14), integrating (1.13) between R and $2R$, we obtain for $2R < 1$

$$\log\left(2^{1-n}\frac{H(2R)}{H(R)}\right) \le 2\log 2N(1). \tag{1.15}$$

Taking exponentials, and integrating in the radial variable R, (1.15) yields

$$\int_0^{2R}\int_{\partial B_\rho} u^2 dH_{n-1} d\rho \le 2^n e^{N(1)\log 4}\int_0^R\int_{\partial B_\rho} u^2 dH_{n-1} d\rho, \tag{1.16}$$

i.e., (1.8) with $C = 2^n e^{N(1)\log 4}$.

In Section 2 we take up the approach outlined above to apply it to the case of solutions to the general equation (1.3). We define appropriate modifications of the functions $H(r)$ and $D(r)$ introduced earlier. These turn out to be the intrinsic analogs of (1.9), (1.10) when $\mathbf{R}^n$ is endowed with a Riemannian metric suitably associated to (1.3). We thus show that (1.14) is a special case of a general principle. In Theorems 2.1 and 2.2 below we prove that if u is a solution to (1.3), then the *modified generalized frequency* (2.13) is a monotone nondecreasing function of the radius r. This allows us to apply the ideas outlined above to the proof of Theorem 1.3.

For our results to hold the assumption (1.1) on $A(x) = (a_{ij}(x))$ plays a basic role. One might wonder if this constitutes an unpleasant limitation of our method and weaker requirements on $A(x)$ be possible. In this respect we emphasize that our results are best possible. In fact, in Section 3 we show that Theorem 1.3 implies Theorem 1.2. Now a well-known counterexample by Plis, [P], shows that if the coefficients of $A(x)$ in (1.3) are only α-Hölder continuous (any $\alpha \in (0,1)$!) then (i) of Theorem 1.2 may fail in general. Therefore, we cannot hope to get (1.8) if we weaken the assumptions of $A(x)$. A unique continuation theorem for divergence form elliptic operators with $C^{0,1}$ coefficients had already been proved in a rather difficult paper by Aronszajn, Krzywicki and Szarski, [AKS], by means of Carleman type inequalities. Our results, therefore, give a new and considerably easier proof of those in [AKS]. One should also see the recent paper by Hörmander, [H], in which unique continuation for operators with singular lower order terms is proved. We wish to mention that Walter Littman has kindly brought to our attention that in his 1965 lecture notes Agmon, [A], had given a proof of unique continuation for operators with C^1 coefficients based on convexity properties of integral averages. Agmon's approach, however, is completely different from ours. It relies on a reduction of the problem to an abstract ode setting and does not seem to extend to operators containing singular zero order terms (see below).

In Section 4 we prove strong unique continuation theorems for some classes of Schrödinger equations. Our point there is to show that our method can be applied to operators containing lower order terms that are allowed to be very singular. Although we restrict our attention to some special classes of singular potentials, it will be clear from the proofs that the method has a wider range of application and similar ideas should work out more in general. We already have some interesting partial results and hope to return to this matter in a subsequent study.

In the unit ball $B_1 \subset \mathbf{R}^n$ we consider solutions, u, of the Schrödinger equation

$$(1.17)\qquad -\Delta u + Vu = 0 \qquad \text{in } B_1,$$

where

$$(1.18)\qquad V(x) = \frac{c}{|x|^m}, \qquad c \in \mathbf{R},\ m \geq 0.$$

(We actually allow more general V's, but see Section 4 for precise details.) Among all the potentials (1.18) there is one that plays the role of a threshold. It is the *inverse square potential* of quantum mechanics, $V(x) = c/|x|^2$. The corresponding Schrödinger operator has received a lot of attention because of the strange featureit displays: it behaves nicely or badly depending on the value of the constant c, see, e.g., [KSWW], [RS], [BG] and [F]. Because of this $c/|x|^2$ is called *strongly singular*. Physically, the inverse square potential arises, for example, in the Hamiltonian for a spin-zero particle in a Coulomb field, see [C].

Our main result concerning unique continuation can be stated as follows. Solutions of (1.17) have the strong unique continuation property (i.e., they cannot vanish of infinite order at one point $x_0 \in B_1$ without being identically zero) so long as $0 \leq m \leq 2$ for V in (1.18). In a counterexample at the end of Section 4 we show that for any $m > 2$ unique continuation fails in general. For brevity's sake we give the proof of the positive results only in the two limit cases, $m = 2$ (Theorem 4.1) and $m = 0$ (Theorem 4.2). However, the case $0 < m < 2$ can be deduced by following the arguments in Theorems 4.1 and 4.2. The idea is to study the natural frequency function for solutions of (1.17), i.e.,

$$(1.19)\qquad N(r) = \frac{rI(r)}{H(r)},$$

where $H(r)$ is as in (1.9) and

$$(1.20)\qquad I(r) = \int_{B_r} (|\nabla u|^2 + Vu^2)dx.$$

It is remarkable that (see the proof of Theorem 4.1) when $V = c/|x|^2$, then $N(r)$ has the same monotone character of the frequency of a harmonic function, see (1.14).

Concerning Theorem 4.1 we mention that recently D. Jerison and C. Kenig have succeeded in proving strong unique continuation for $-\Delta + V$ under the assumption $V \in L^{n/2}_{\text{loc}}(\mathbf{R}^n)$. Their proof is based on a difficult $L^p - L^q$ Carleman type inequality, involving sharp exponents, see [JK]. However, since $c/|x|^2 \notin L^{n/2}_{\text{loc}}(\mathbf{R}^n)$ our result does not follow from those in [JK]. Moreover, since we do not require any local restriction on the size of V, Theorem 4.1 is not contained either in E. Stein's subsequent improvement of Jerison and Kenig's results concerning $V \in L^{n/2,\infty}_{\text{loc}}(\mathbf{R}^n)$, the Lorentz space of weak $n/2$ type, see [S]. We should also remark that, suitably modified, the proof of Theorem 4.1 can be used to conclude unique continuation for any radial potential that in B_1 is bounded by $c/|x|^2$, for some $c \in \mathbf{R}^+$.

Finally, we wish to thank Eugene Fabes for pointing out to us that, when dealing with the Euclidean Laplacian, the radial deformation of the proof of Theorem 2.1 can be replaced by a (equivalent) classical identity of Rellich.

2. First variation and monotonicity property. Let $B_1 = \{x \in \mathbf{R}^n : |x| < 1\}$ denote the open unit ball in $\mathbf{R}^n$, $n \geq 3$. We assume that on B_1 a Lipschitz metric tensor is given, $g_{ij}(x)dx_i \otimes dx_j$, which in polar coordinates $(r,\theta_1,\ldots,\theta_{n-1})$ takes the form

$$dr \otimes dr + r^2 b_{ij}(r,\theta) d\theta_i \otimes d\theta_j. \tag{2.1}$$

Here we suppose that

$$b_{ij}(0,0) = \delta_{ij}, \qquad i,j = 1,2,\ldots,n-1; \tag{2.2}$$

and that there exists a positive constant Λ such that

$$\left| \frac{\partial}{\partial r} b_{ij}(r,\theta) \right| \leq \Lambda, \qquad i,j = 1,2,\ldots,n-1. \tag{2.3}$$

We denote by $g^{ij}(x)$ the elements of the inverse matrix of $g_{ij}(x)$, and we set

$$g(x) = |\det(g_{ij}(x))|. \tag{2.4}$$

With the notations $\nabla_M u$ and $\mathrm{div}_M X$ we denote, respectively, the intrinsic gradient of a function u and the intrinsic divergence of a vector field X on B_1 in the metric $g_{ij}dx_i \otimes dx_j$. We recall that (see, e.g., [He], p. 387)

$$\nabla_M u = \sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j} \tag{2.5}$$

$$\mathrm{div}_M X = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (\sqrt{g} X_i).$$

Let μ be a Lipschitz function on $\bar{B}_1$ such that there exist two constants $C_1, C_2 > 0$ for which

$$C_1 \leq \mu(x) \leq C_2, \qquad x \in \bar{B}_1. \tag{2.6}$$

We assume that there exists a $\Lambda > 0$ such that in polar coordinates on B_1 we have

$$\mu(0,0) = 1, \qquad \left| \frac{\partial}{\partial r} \mu \right| \leq \Lambda. \tag{2.7}$$

We consider a solution $u \in H^{1,2}(B_1)$ of the equation

$$\mathrm{div}_M(\mu(x) \nabla_M u(x)) = 0 \qquad \text{in } B_1, \tag{2.8}$$

and for $r \in (0,1)$ we introduce the quantities

$$(2.9)\qquad D(r)=\int_{B_r}\mu|\nabla_M u|^2 dV_M,$$

$$(2.10)\qquad H(r)=\int_{\partial B_r}\mu u^2 dV_{\partial B_r}.$$

Here B_r represents the geodesic ball in the metric g of radius r and center at the origin. By (2.1) B_r coincides with the usual Euclidean ball. If we set $b(r,\theta) = |\det(b_{ij}(r,\theta))|$, and observe that $\sqrt{g(r,\theta)} = r^{n-1}\sqrt{b(r,\theta)}$, we can rewrite (2.10) as

$$(2.11)\qquad H(r)=r^{n-1}\int_{\partial B_1}\mu(r,\theta)u^2(r,\theta)\sqrt{b(r,\theta)}\,d\theta.$$

For $r \in (0,1)$ we define what we call the *generalized frequency* of u as

$$(2.12)\qquad N(r)=\frac{rD(r)}{H(r)}\qquad \text{if } H(r)\neq 0.$$

The main result in this section is the following

Theorem 2.1. *If $u \in H^{1,2}(B_1)$ is a nontrivial solution of (2.8), then there exists a positive constant $C = C(n,\Lambda)$ such that*

$$(2.13)\qquad \bar{N}(r)=\exp(Cr)N(r)$$

is a monotone nondecreasing function of $r \in (0,1)$.

Proof. We start by considering $N(r) = rD(r)/H(r)$. Differentiation gives

$$(2.14)\qquad N'(r)=N(r)\left[\frac{D'(r)}{D(r)}+\frac{1}{r}-\frac{H'(r)}{H(r)}\right].$$

Therefore, the theorem will be proved if we can show that

$$(2.15)\qquad \frac{D'(r)}{D(r)}+\frac{1}{r}-\frac{H'(r)}{H(r)}\geq -C(n,\Lambda).$$

To this end we compute the derivatives $H'(r)$ and $D'(r)$. By (2.11) we obtain

$$(2.16)\qquad H'(r)=\frac{n-1}{r}H(r)+\int_{\partial B_r}\frac{1}{\sqrt{b}}\frac{\partial}{\partial\rho}(\mu\sqrt{b})u^2 dV_{\partial B_r}+2\int_{\partial B_r}\mu u u_\rho dV_{\partial B_r},$$

where u_ρ denotes radial differentiation, i.e., $u_\rho = \langle\nabla_M u, x/\rho\rangle$. Using (2.3), (2.6) and (2.7) we can rewrite (2.16) as

$$(2.17)\qquad H'(r)=\left[\frac{n-1}{r}+O(1)\right]H(r)+2\int_{\partial B_r}\mu u u_\rho dV_{\partial B_r},$$

where $O(1)$ denotes a function of (r,θ) which is bounded in absolute value by a constant $C = C(n,\Lambda)$. Next we observe that because of (2.8) we have

$$\int_{B_r} \operatorname{div}_M(\mu\nabla_M(u^2))dV_M = 2\int_{B_r} \mu|\nabla_M u|^2 dV_M. \tag{2.18}$$

On the other hand the divergence theorem gives

$$\int_{B_r} \operatorname{div}_M(\mu\nabla_M(u^2))dV_M = 2\int_{\partial B_r} \mu u u_\rho dV_{\partial B_r}. \tag{2.19}$$

Using these observations, we can rewrite (2.17) as

$$H'(r) = \left[\frac{n-1}{r} + O(1)\right]H(r) + 2D(r). \tag{2.20}$$

We now turn to the computation of $D'(r)$.

Step I. A radial deformation. For $0 < r$, $\Delta r < 1/2$ fixed, we define the map $w(t) = w(t;r,\Delta r):\mathbf{R}^n \to \mathbf{R}^+$ by

$$w(t)(x) = \begin{cases} t, & \text{if } \rho(x) \le r \\ 1, & \text{if } \rho(x) \ge r + \Delta r \\ t\dfrac{r+\Delta r - \rho(x)}{\Delta r} + \dfrac{\rho(x) - r}{\Delta r}, & \text{if } r \le \rho(x) \le r + \Delta r \end{cases} \tag{2.21}$$

where we recall that by (2.7) we have $\rho(x) = \operatorname{dist}_M(0,x) = |x|$. Now we define a map $\ell(t):\mathbf{R}^n \to \mathbf{R}^n$, for $0 < t < 1 + \Delta r/(r + \Delta r)$, by

$$\ell(t)(x) = w(t)(x)x. \tag{2.22}$$

It is easy to check that $\ell(t)$ is bi-Lipschitz; therefore if for $u \in H^{1,2}(B_1)$ we set $u^t(x) = u(\ell(t)^{-1}(x))$, then $u^t \in H^{1,2}(B_1)$.

Step II. First variation estimates. Since u is a solution of (2.8) we have

$$\frac{d}{dt} I[u^t]\big|_{t=1} = 0, \tag{2.23}$$

where we have set

$$\begin{aligned} I[u^t] &= \int_{B_1} \mu|\nabla_M u^t|^2 dV_M \\ &= \int_{B_{rt}} \mu|\nabla_M u^t|^2 dV_M + \int_{B_{r+\Delta r}\setminus B_{rt}} \mu|\nabla_M u^t|^2 dV_M + \int_{B_1\setminus B_{r+\Delta r}} \mu|\nabla_M u^t|^2 dV_M \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{2.24}$$

Now

$$I_3 = \int_{B_1\setminus B_{r+\Delta r}} \mu|\nabla_M u^t|^2 dV_M = \int_{B_1\setminus B_{r+\Delta r}} \mu|\nabla_M u|^2 dV_M, \tag{2.25}$$

so that

$$\left.\frac{dI_3}{dt}\right|_{t=1} = 0.$$

Next, we have

$$(2.26)\qquad \begin{aligned} I_1 &= \int_{B_r} \mu|\nabla_M u^t|^2 dV_M \\ &= \int_0^r \int_{S^{n-1}} \mu(t\rho,\theta)u_\rho^2(\rho,\theta)\frac{\sqrt{g(t\rho,\theta)}}{t}\,d\theta d\rho \\ &\quad + \int_0^r \int_{S^{n-1}} \mu(t\rho,\theta)u_{\theta_i}(\rho,\theta)u_{\theta_j}(\rho,\theta)b^{ij}(t\rho,\theta)t\sqrt{g(t\rho,\theta)}\,d\theta d\rho, \end{aligned}$$

where in the second term on the right-hand side of (2.26) we have used the summation convention and we have denoted by b^{ij} the entries of the matrix $(b_{ij})^{-1}$. We remark that by (2.7)

$$\left|\frac{d}{dt}\mu(t\rho,\theta)\right| \le \Lambda\rho.$$

Moreover, if we set $1 + \varepsilon(\rho,\theta) = \sqrt{b(\rho,\theta)}$, we can write

$$(2.27)\qquad \begin{cases} \sqrt{g(t\rho,\theta)} = t^{n-1}\rho^{n-1}[1 + \varepsilon(t\rho,\theta)] \\ b^{ij}(t\rho,\theta)\sqrt{g(t\rho,\theta)} = t^{n-3}\rho^{n-3}[\delta_{ij} + \overline{\varepsilon_{ij}}(t\rho,\theta)] \end{cases}$$

for certain $\overline{\varepsilon_{ij}}(t\rho,\theta)$. Because of (2.3), we have

$$(2.28)\qquad \left|\frac{\partial}{\partial\rho}\varepsilon(\rho,\theta)\right| \le C(n,\Lambda), \qquad \left|\frac{\partial}{\partial\rho}\overline{\varepsilon_{ij}}(\rho,\theta)\right| \le C(n,\Lambda),$$

therefore (2.26) gives

$$(2.29)\qquad \left.\frac{dI_1}{dt}\right|_{t=1} = (n-2)D(r) + O(r)D(r).$$

In (2.29) $O(r)$ denotes a function of (r,θ) whose absolute value is bounded by Cr, where $C = C(n,\Lambda)$.

Finally, we wish to estimate $dI_2/dt|_{t=1}$. To this end, we need to introduce some notation. We set

$$\gamma(t)(\rho) = w(t)(x)\rho = \rho\left[t\frac{r+\Delta r-\rho}{\Delta r} + \frac{\rho - r}{\Delta r}\right]$$

and

$$h^{-1}(t,\rho) = \frac{d}{d\rho}\gamma(t)(\rho) = \frac{\rho(1-t) + t(r+\Delta r-\rho) + (\rho - r)}{\Delta r}.$$

Then

$$(2.30)\quad I_2 = \int_{B_{r+\Delta r}\setminus B_{rt}} \mu|\nabla_M u^t|^2 dV_M$$

$$+ \int_r^{r+\Delta r}\int_{S^{n-1}} \mu(\gamma(t)(\rho),\theta)h^2(\rho,t)u_\rho^2(\rho,\theta)\sqrt{g(\gamma(t)(\rho),\theta)}d\theta d(\gamma(t)\rho)$$

$$+ \int_r^{r+\Delta r}\int_{S^{n-1}} \mu(\gamma(t)(\rho),\theta)u_{\theta_i}(\rho,\theta)u_{\theta_j}(\rho,\theta)b^{ij}(\gamma(t)(\rho),\theta)\sqrt{g(\gamma(t)(\rho),\theta)}d\theta d(\gamma(t)\rho).$$

Using expansions similar to (2.27) for $\sqrt{g(\gamma(t)(\rho),\theta)}$ and

$$b^{ij}(\gamma(t)(\rho),\theta)\sqrt{g(\gamma(t)(\rho),\theta)}$$

we have from (2.30)

$$(2.31)\quad \left.\frac{dI_2}{dt}\right|_{t=1} = \int_{B_{r+\Delta r}\setminus B_r} \mu\left[((n-1)+O(\rho))\frac{r+\Delta r-\rho}{\Delta r} - \frac{r+\Delta r-2\rho}{\Delta r}\right]u_\rho^2 dV_M$$

$$+ \int_{B_{r+\Delta r}\setminus B_r} \mu\left[(n-3)\frac{r+\Delta r-\rho}{\Delta r}(1+O(\rho)) + \frac{r+\Delta r-2\rho}{\Delta r}\right](|\nabla_M u|^2 - u_\rho^2)dV_M$$

$$+ \int_{B_{r+\Delta r}\setminus B_r} O(1)|\nabla_M u|^2 dV_M.$$

Now we let $\Delta r \to 0^+$ in (2.31) obtaining

$$(2.3)\qquad \left.\frac{dI_2}{dt}\right|_{t=1} = 2r\int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r} - r\int_{\partial B_r} \mu|\nabla_M u|^2 dV_{\partial B_r}.$$

From (2.23), (2.25), (2.29) and (2.32) we finally obtain

$$(2.33)\qquad rD'(r) - ((n-2)+O(r))D(r) = 2r\int_{\partial B_r} \mu u_\rho^2 dV_{\partial B_r}.$$

This formula is the basic tool in the proof of Theorem 2.1. By it and (2.20) we can finally estimate the term within brackets in (2.14). We have

$$(2.34)\quad \frac{D'(r)}{D(r)} + \frac{1}{r} - \frac{H'(r)}{H(r)} = O(1) + 2\frac{\int_{\partial B_r}\mu u_\rho^2 dV_{\partial B_r}}{\int_{\partial B_r}\mu u u_\rho dV_{\partial B_r}} - 2\frac{\int_{\partial B_r}\mu u u_\rho dV_{\partial B_r}}{\int_{\partial B_r}\mu u^2 dV_{\partial B_r}}$$

$$\geq O(1) \geq -C(n,\Lambda).$$

The second-to-the-last inequality in (2.34) is a consequence of Schwarz' inequality. (2.34) is what we need to complete the proof of the theorem. From it and

(2.14) it follows that $\exp(C(n,\Lambda)r)N(r) = \bar{N}(r)$ is a monotone nondecreasing function on $(0,1)$.

Remark. When $A(x) =$ Identity, i.e., when $L = \Delta$, then the computation of $D'(r)$ can be carried out more directly by using a classical formula of Rellich. In fact, the radial deformation (2.21) in this case is equivalent to Rellich's formula. We will use Rellich's formula in Section 4 to study the frequency of Schrödinger operators.

We now wish to derive a consequence of Theorem 2.1 that will be relevant for the applications in Section 3. Let $L = \operatorname{div}(A(x)\nabla)$ be an elliptic operator in $\Omega \supset \bar{B}_1$ satisfying (1.1) and (1.2). For $n \geq 3$ we define a Lipschitz metric on $B_1 \subset \mathbf{R}^n$, $g_{ij}(x)dx_i \otimes dx_j$ by setting

$$g_{ij}(x) = a^{ij}(x)(\det A)^{1/(n-2)}. \tag{2.35}$$

In (2.35) a^{ij} denote the entries of $A(x)^{-1}$. We have

(2.36) *$Lu = 0$ in B_1 if and only if* $\operatorname{div}_M(\nabla_M u) = 0$ *in the metric given by (2.35).*

Letting $(g^{ij}(x)) = (g_{ij}(x))^{-1}$ and setting

$$r(x)^2 = g_{ij}(0)x_ix_j, \qquad \eta(x) = g^{k\ell}(x)\frac{\partial r}{\partial x_k}(x)\frac{\partial r}{\partial x_\ell}(x), \tag{2.37}$$

we introduce a new metric $\bar{g}_{ij}(x)dx_i \otimes dx_j$ in B_1 by defining

$$\overline{g_{ij}}(x) = \eta(x)g_{ij}(x). \tag{2.38}$$

One easily verifies that η is a Lipschitz function on B_1, whose Lipschitz constant depends on the Lipschitz constant, Γ, of a_{ij}, see (1.1). Moreover, as proved in Section 3 of [AKS], in the intrinsic geodesic polar coordinates with pole at zero, the metric tensor $\bar{g}_{ij}dx_i \otimes dx_j$ takes the form $dr \otimes dr + r^2 b_{ij}(r,\theta)d\theta_i \otimes d\theta_j$, where b_{ij} satisfy (2.2) and (2.3) with a Λ depending on Γ, λ and n. In this new metric we can rewrite (2.36) as

$$\operatorname{div}_{\bar{M}}(\mu(x)\nabla_{\bar{M}}u(x)) = 0 \tag{2.39}$$

where μ is a Lipschitz function on B_1 satisfying (2.6), (2.7) with C_1, C_2 and Λ depending on Γ, λ, n.

By the above reductions and Theorem 2.1 we then obtain

Theorem 2.2. *Let Ω be a connected open subset of $\mathbf{R}^n$, $n \geq 3$, such that $\Omega \supset \bar{B}_1$, and let $L = \operatorname{div}(A(x)\nabla)$ be an elliptic operator on Ω with $A(x)$ satisfying (1.1) and (1.2). Then there exists a positive constant $C = C(n,\lambda,\Gamma)$ such that, if $N(r)$ is defined as in (2.12), relative to the metric given by (2.38), then*

(2.40) *$\bar{N}(r) = \exp(Cr)N(r)$ is monotone nondecreasing in $r \in (0,1)$.*

3. A_p weights and unique continuation. In this section we apply the results of Section 2 to provide the proofs of Theorems 1.1, 1.2, and 1.3. We begin with the

Proof of Theorem 1.3. Let $x_0 \in B_1$, the unit ball centered at the origin, and let $B_R(x_0)$ and $B_{2R}(x_0)$ two concentric balls centered at x_0, radii respectively R and $2R$, such that $B_{2R}(x_0) \subset B_1$. We wish to prove that (1.8) holds for $B_R(x_0)$ and $B_{2R}(x_0)$, with a constant depending only on u, Γ, λ and n. We are going to use Theorem 2.2. We recall that (2.40) holds relative to balls centered at the origin. However, it is clear that the same conclusion is true, with a uniform constant $C = C(n,\lambda,\Gamma)$, if we consider frequencies relative to balls centered at any point $x_0 \in B_1$ (we recall that we are assuming $\bar{B}_2 \subset \Omega$). Hence, without loss of generality we may suppose that $x_0 = 0$, and we will write $B_r(x_0) = B_r$, for any $r \in (0,1)$.

If μ is the Lipschitz function in (2.39) we set as in Theorem 2.1

$$H(r) = \int_{\partial B_r} \mu u^2 dV_{\partial B_r}.$$

From (2.20) we get

$$\left(\log \frac{H(r)}{r^{n-1}}\right)' = O(1) + 2\bar{N}(r)\frac{\exp(-Cr)}{r}, \tag{3.1}$$

where we have used the definition (2.13) of the modified frequency, and $C = C(u,\lambda,\Gamma)$ is the constant appearing in (2.40). Now for any $R \in (0,1)$ such that $2R < 1$ we integrate (3.1) between R and $2R$. Using (2.40) we obtain

$$\log\left(\frac{H(2R)}{H(R)}2^{n-1}\right) \le C'R + 2\log 2\bar{N}(1). \tag{3.2}$$

Exponentiating (3.2) yields

$$H(2R) \le H(R)[2^{n-1}\exp(C' + 2\log 2\bar{N}(1))]. \tag{3.3}$$

Integrating (3.3) in R gives

$$\int_{B_{2R}} \mu u^2 dV_{B_{2R}} \le C \int_{B_R} \mu u^2 dV_{B_R}. \tag{3.4}$$

Finally, using the bounds (2.6) on μ, we get from (3.4)

$$\int_{B_{2R}} u^2 dx \le C \int_{B_R} u^2 dx \tag{3.5}$$

which is what we wanted to prove. The constant C in (3.5) has the desired dependence.

We now turn to the

Proof of Theorem 1.2. *Part (i).* Without loss of generality we may assume that u vanishes of infinite order at $x_0 = 0$, i.e., (1.7) holds for all $R \le R_0$. We wish to show that $u \equiv 0$ in B_{R_0}. The following argument is known, see, e.g., [G], p. 135. We reproduce it for the sake of completeness. By (3.5) we obtain

$$(3.6)\qquad \int_{B_{R_0}} u^2 dx \leq C^k \int_{B_{R_0 2^{-k}}} u^2 dx = C^k |B_{R_0 2^{-k}}|^\alpha \frac{1}{|B_{R_0 2^{-k}}|^\alpha} \int_{B_{R_0 2^{-k}}} u^2 dx$$

with $\alpha > 0$ to be chosen. Now we let α be such that $C2^{-n\alpha} = 1$, where C is the constant in (3.6). This yields

$$(3.7)\qquad \int_{B_{R_0}} u^2 dx \leq (\omega_n R_0^n)^\alpha \frac{1}{|B_{R_0 2^{-k}}|^\alpha} \int_{B_{R_0 2^{-k}}} u^2 dx \to 0 \qquad \text{as } k \to \infty$$

because of (1.7).

Part (ii). The proof of this part relies on (ii) of Theorem 1.1, which we still must prove. However, let us assume that Theorem 1.1 is true. Hence, if u is a nonconstant solution of (1.3) a $q > 1$ and a $B > 0$ exist (depending only on u, Γ, λ and n) such that (1.5) holds. By the results in [CF] there exists a $B' > 0$, depending on q and B in (1.5), such that for any B_R, such that $B_{2R} \subset B_1$,

$$(3.8)\qquad \int_{B_{2R}} |\nabla u| dx \leq B' \int_{B_R} |\nabla u| dx.$$

It is now clear that, since we are assuming $u \not\equiv$ const., $|\nabla u|$ cannot vanish of infinite order at any point $x_0 \in B_1$. This completes the proof of Theorem 1.2, modulo the

Proof of Theorem 1.1. To begin with, we recall a well-known local estimate that holds for solutions of even more general equations than (1.3) (see [M] for instance): there exists a constant $\gamma = \gamma(\lambda, n)$ such that for any ball B_R such that $B_{2R} \subset B_1$ we have

$$(3.9)\qquad \sup_{B_R} u^2 \leq \gamma \frac{1}{|B_{2R}|} \int_{B_{2R}} u^2 dx.$$

On the other hand, for every $\delta > 0$ we trivially have

$$(3.10)\qquad \left(\frac{1}{|B_R|} \int_{B_R} u^{2(1+\delta)} dx\right)^{1/(1+\delta)} \leq \sup_{B_R} u^2.$$

By (3.9), (3.10) and (1.8) of Theorem 1.3 we obtain for every $\delta > 0$

$$(3.11)\qquad \left(\frac{1}{|B_R|} \int_{B_R} u^{2(1+\delta)} dx\right)^{1/(1+\delta)} \leq C \frac{1}{|B_R|} \int_{B_R} u^2 dx,$$

i.e., u^2 satisfies a reverse Hölder inequality. The results in [CF] assure the existence of some $r \in (1, \infty)$ and $A' > 0$ such that

$$(3.12)\qquad \left(\frac{1}{|B_R|} \int_{B_R} u^2 dx\right) \left(\frac{1}{|B_R|} \int_{B_R} (u^2)^{-1/(r-1)} dx\right)^{r-1} \leq A',$$

for every ball B_R, with $B_{2R} \subset B_1$. A' and r depend on C in (3.11) and the choice of a $\delta > 0$. Taking $p = (r + 1)/2 > 1$, and $A = \sqrt{A'}$ in (3.12), by Schwarz' inequality we obtain (1.4).

Part (ii). Let B_R be a ball in B_1, such that $B_{2R} \subset B_1$. If we set

$$u_R = \frac{1}{|B_R|} \int_{B_R} u\,dx,$$

then $u - u_R$ too is a solution to (1.3) in Ω. The assumption $u \not\equiv$ const., and the maximum principle, imply that

$$\inf_{\substack{0<R<1 \\ x_0 \in \bar{B}_1}} \int_{\partial B_1(x_0)} |u - u_R|^2 dH_{n-1} = \mu > 0, \tag{3.13}$$

where we have denoted with $B_1(x_0)$ the ball of radius 1 and center at x_0. (3.13) assures that in applying Theorem 1.3 to $u - u_R$ the constant in (1.8), which involves the modified frequency (2.13) of $u - u_R$, is independent of u_R, for any $B_R \subset B_{2R} \subset B_1$. Therefore, by Theorem 1.3 we obtain

$$\int_{B_{2R}} |u - u_R|^2 dx \leq C \int_{B_R} |u - u_R|^2 dx, \tag{3.14}$$

where C has the stated dependence. Now Caccioppoli's inequality (see [M]) applied to $u - u_R$ gives

$$\int_{B_R} |\nabla u|^2 dx \leq C/R^2 \int_{B_{2R}} |u - u_R|^2, \tag{3.15}$$

with a $C = C(\lambda, n)$. On the other hand, Poincaré's inequality yields

$$\frac{1}{|B_R|} \int_{B_R} |u - u_R|^2 \leq CR^2 \left(\frac{1}{|B_R|} \int_{B_R} |\nabla u|^s dx \right)^{2/s}, \tag{3.16}$$

with $s = 2n/(n + 2) < 2$. By (3.15), (3.14) and (3.16) we finally obtain

$$\left(\frac{1}{|B_R|} \int_{B_R} |\nabla u|^2 dx \right)^{1/2} \leq C \left(\frac{1}{|B_R|} \int_{B_R} |\nabla u|^s dx \right)^{1/s}, \tag{3.17}$$

i.e., a reverse Hölder inequality for $|\nabla u|^s$. To obtain (1.5) from (3.17) now requires an argument completely similar to the one given for $|u|$ in the proof of Part (i), and we omit the details. The proof is completed.

Remark. We wish to emphasize that (3.17) implies higher integrability of $|\nabla u|$. This is a consequence of the self-improving feature ($A_p \Rightarrow A_{p-\varepsilon}$ for some $\varepsilon > 0$) that is inherent to A_p weights, see [CF]. For an interesting account of the connection between reverse Hölder inequalities and L^p properties of solutions of elliptic equations and systems, cf. [G].

4. Unique continuation for a class of Schrödinger operators. The purpose of this section is to show that the ideas of Sections 2 and 3 can be applied to study unique continuation for solutions to equations with singular lower order terms. Let $f: \mathbf{R}^n \to \mathbf{R}$ be a bounded function homogeneous of degree zero, i.e., if $\omega = x/|x|$, $x \neq 0$, we have

$$f(x) = f(\omega), \quad \text{and} \quad |f(\omega)| \leq C \qquad \text{for all } \omega \in S^{n-1}. \tag{4.1}$$

We consider the Schrödinger operator

$$H = -\Delta + \frac{f}{|x|^2}. \tag{4.2}$$

Our aim is to provide a simple proof of the following

Theorem 4.1. *Let $u \in H^{1,2}_{\text{loc}}(B_1)$ be a solution in B_1 of $Hu = 0$. Then u cannot vanish of infinite order at one point $x_0 \in B_1$, unless $u \equiv 0$ in B_1.*

The proof of the theorem will be given below. As mentioned in Section 1, since $f/|x|^2 \notin L^{n/2}_{\text{loc}}(\mathbf{R}^n)$, and moreover we do not require any restriction on the size of f, Theorem 4.1 is neither included in the general result in [JK], nor in the sharper result in [S]. We mention that a simple proof of the strong unique continuation property for nonnegative solutions of general Schrödinger operators with potentials $V \in L^{n/2,\infty}_{\text{loc}}$ has been given in [CG].

The following theorem will be needed as a lemma to Theorem 4.1. However, it has an interest in its own right since it provides a new proof of the results in [Ca] and [Mu] that does not use Carleman estimates.

Theorem 4.2. *Let $u \in H^{1,2}_{\text{loc}}(B_1)$ be a solution to the equation*

$$-\Delta u + Vu = 0 \qquad \text{in } B_1, \tag{4.3}$$

where $V \in L^{\infty}_{\text{loc}}(B_1)$. If u vanishes of infinite order at $x_0 \in B_1$, then $u \equiv 0$ in B_1.

Proof. Without loss of generality we can assume that $x_0 = 0$, and prove that there exists a small ball centered at the origin, B_ε, such that $u \equiv 0$ in B_ε. Following the ideas of Section 3, for every $r \in (0,1)$ we define the quantities

$$H(r) = \int_{\partial B_r} u^2 dH_{n-1}, \qquad I(r) = \int_{B_r} (|\nabla u|^2 + Vu^2)dx. \tag{4.4}$$

We remark that since u is a solution of (4.3), u minimizes $I(r)$. Moreover, since $\Delta(u^2) = 2(|\nabla u|^2 + Vu^2)$, the divergence theorem gives

$$I(r) = \int_{\partial B_r} uu_\rho dH_{n-1}, \tag{4.5}$$

where $u_\rho = \langle \nabla u, x/\rho \rangle$, $|x| = \rho$. Differentiating $H(r)$ we obtain

$$H'(r) = \frac{n-1}{r} H(r) + 2\int_{\partial B_r} uu_\rho dH_{n-1} = \frac{n-1}{r} H(r) + 2I(r). \tag{4.6}$$

We now define the following *frequency* function

$$N(r) = \frac{rI(r)}{H(r)} \qquad 0 < r < 1. \tag{4.7}$$

Following the arguments in the proof of Theorem 1.3, it is clear that if we could show that there exists an $r_0 \in (0,1)$ such that $N \in L^\infty(0,r_0)$, then we would be done. In fact, from this and equation (4.6) we would obtain the doubling condition for $H(r)$ and hence, by integration, a solid doubling condition like (1.8), at least for small r's. Then $u \equiv 0$ in B_{r_0} would follow.

In order to proceed we compute the derivative of $I(r)$. Since we are dealing with the Euclidean Laplacian we take an approach that, apparently different from the radial deformation of Theorem 2.1, is in fact equivalent to it: we use a classical identity due to Rellich. The use of such an identity was suggested to us by Eugene Fabes. We then have

$$\begin{aligned} I'(r) &= \int_{\partial B_r} |\nabla u|^2 dH_{n-1} + \int_{\partial B_r} Vu^2 dH_{n-1} \\ &= \frac{1}{r}\int_{\partial B_r} \langle x|\nabla u|^2, x/r\rangle dH_{n-1} + \int_{\partial B_r} Vu^2 dH_{n-1} \\ &= \frac{n}{r}\int_{B_r} |\nabla u|^2 dx + \frac{2}{r}\sum_{i,j=1}^{n}\int_{B_r} x_i u_{x_j} u_{x_i x_j} dx + \int_{\partial B_r} Vu^2 dH_{n-1}, \end{aligned} \tag{4.8}$$

the last equality being a consequence of the divergence theorem. Integrating by parts the second term in the right-hand side of (4.8) and using the fact that $\Delta u = Vu$ gives

$$\begin{aligned} I'(r) = \frac{n-2}{r}\int_{B_r} |\nabla u|^2 dx + 2\int_{\partial B_r} u_\rho^2 dH_{n-1} \\ + \int_{\partial B_r} Vu^2 dH_{n-1} - \frac{2}{r}\int_{B_r} \langle x, \nabla u\rangle Vu dx. \end{aligned} \tag{4.9}$$

(4.9) can be rewritten

$$\begin{aligned} I'(r) = \frac{n-2}{r} I(r) + 2\int_{\partial B_r} u_\rho^2 dH_{n-1} + \int_{\partial B_r} Vu^2 dH_{n-1} \\ - \frac{n-2}{r}\int_{B_r} Vu^2 dx - \frac{2}{r}\int_{B_r} \langle x, \nabla u\rangle Vu dx. \end{aligned} \tag{4.10}$$

We now

Claim. There exists an $r_0 \in (0,1)$, depending on $\|V\|_{L^\infty}$, such that for every $r \in (0,r_0)$ we have

$$\int_{B_r} u^2 dx \le r \int_{\partial B_r} u^2 dH_{n-1}. \tag{4.11}$$

Proof of the claim. For every $r \in (0,1)$ an integration by parts gives

$$\int_{B_r} \Delta(u^2)(r^2 - |x|^2)dx = 2r \int_{\partial B_r} u^2 dH_{n-1} - 2n \int_{B_r} u^2 dx. \tag{4.12}$$

On the other hand

$$\int_{B_r} \Delta(u^2)(r^2 - |x|^2)dx = 2 \int_{B_r} (|\nabla u|^2 + Vu^2)(r^2 - |x|^2)dx. \tag{4.13}$$

(4.12) and (4.13) yield

$$r \int_{\partial B_r} u^2 dH_{n-1} \ge \int_{B_r} (n + V(r^2 - |x|^2))u^2 dx. \tag{4.14}$$

We now set $\|V\|_{L^\infty} = \|V\|_{L^\infty(B_{1/2})}$ and choose $r_0 \in (0,1/2)$ such that

$$r_0^2 \le \frac{n-1}{\|V\|_{L^\infty}}. \tag{4.15}$$

For all $r \in (0,r_0)$ and $|x| < r$ we then have

$$n + V(r^2 - |x|^2) \ge n - \|V\|_{L^\infty} r^2 \ge n - \|V\|_{L^\infty} r_0^2 \ge 1. \tag{4.16}$$

From (4.16) and (4.14), (4.11) follows.

Next we observe that without loss of generality that we can assume that there exists a small $\bar{r} \in (0,1)$ such that

$$H(r) \neq 0 \qquad \text{for any } r \in (0,\bar{r}). \tag{4.17}$$

Restricting, if needed, the interval $(0,\bar{r})$, we can assume that $\bar{r} = r_0$, where r_0 is determined by (4.15). It follows that $N(r)$ is continuous on $(0,r_0)$. Therefore, if we set

$$\Omega_{r_0} = \{r \in (0,r_0) : N(r) > \max(N(r_0),1)\} \tag{4.18}$$

by a well-known real variable decomposition, see, e.g., [N], p. 48, we have

$$\Omega_{r_0} = \bigcup_{j=1}^{\infty} (a_j,b_j), \qquad a_j,b_j \notin \Omega_{r_0}. \tag{4.19}$$

Now we compute $N'(r)$. By (4.6) and (4.10) we obtain

$$N'(r) = N(r)\left[\frac{I'(r)}{I(r)} + \frac{1}{r} - \frac{H'(r)}{H(r)}\right] \tag{4.20}$$

$$= N(r)\left\{ 2\frac{\int_{\partial B_r} u_\rho^2}{\int_{\partial B_r} uu_\rho} - 2\frac{\int_{\partial B_r} uu_\rho}{\int_{\partial B_r} u^2} + \frac{\int_{\partial B_r} Vu^2 - \frac{n-2}{r}\int_{B_r} Vu^2 - \frac{2}{r}\int_{B_r} \langle x,\nabla u\rangle Vu}{I(r)} \right\}$$

where in the above integrals we have omitted the measures dH_{n-1} and dx for the sake of brevity. Now we pick an arbitrary interval (a_{j_0},b_{j_0}) entering in (4.19) and estimate $N'(r)$ on it in a uniform fashion. Since $N(r) > 1$ for $r \in (a_{j_0},b_{j_0})$, recalling (4.5) and using Schwarz' inequality we see that on (a_{j_0},b_{j_0})

$$\frac{\int_{\partial B_r} u_\rho^2}{\int_{\partial B_r} uu_\rho} - \frac{\int_{\partial B_r} uu_\rho}{\int_{\partial B_r} u^2} \geq 0. \tag{4.21}$$

We now look at the last ratio in (4.20).

In what follows we will denote with $\|V\|_{L^\infty}$ the L^∞ norm of V on B_{r_0}. Since $N(r) > 1$ yields $H(r)/I(r) < r$ we have

$$\left|\frac{\int_{\partial B_r} Vu^2}{I(r)}\right| \leq \|V\|_{L^\infty}\frac{H(r)}{I(r)} < r\|V\|_{L^\infty}. \tag{4.22}$$

By claim (4.11) we get

$$\left|\frac{\frac{n-2}{r}\int_{B_r} Vu^2}{I(r)}\right| \leq \frac{n-2}{r}\|V\|_{L^\infty}\frac{\int_{B_r} u^2}{I(r)} \leq (n-2)r\|V\|_{L^\infty}. \tag{4.23}$$

Finally,

$$\left|\frac{\frac{2}{r}\int_{B_r} \langle x,\nabla u\rangle Vu}{I(r)}\right| \leq 2\|V\|_{L^\infty}\frac{\int_{B_r} |\nabla u||u|}{I(r)} \tag{4.24}$$

$$\leq \|V\|_{L^\infty}\frac{\int_{B_r} (|\nabla u|^2 + Vu^2) - \int_{B_r} Vu^2 + \int_{B_r} u^2}{I(r)}$$

$$\leq 2\|V\|_{L^\infty}(1 + \|V\|_{L^\infty}r^2 + r).$$

From (4.20), (4.21), (4.22), (4.23) and (4.24) we get the existence of a constant $\Lambda = \Lambda(n,\|V\|_{L^\infty},r_0)$ such that for every $r \in (a_{j_0},b_{j_0})$

(4.25) $$N'(r) \geq -N(r)\Lambda.$$

(4.25) implies that

(4.26) $\exp(\Lambda r)N(r)$ is monotone increasing on (a_{j_0}, b_{j_0}).

Since Λ does not depend on the chosen (a_{j_0}, b_{j_0}) and

$$\exp(\Lambda r)N(r) \leq \exp(\Lambda b_{j_0})N(b_{j_0}) \leq \exp(\Lambda r_0)\max(N(r_0), 1)$$

we conclude that $\exp(\Lambda r)N(r)$, hence $N(r)$ is bounded on $(0, r_0)$. From (4.6), as in the proof of Theorem 1.3, we conclude that u satisfies (3.5) for all balls B_R such that $2R < r_0$. This finishes the proof of the theorem.

We are now ready to give the

Proof of Theorem 4.1. Since for $x \neq 0$ we have $V = f(\omega)/|x|^2 \in L^\infty_{\mathrm{loc}}(B_1 \setminus \{0\})$, strong unique continuation at any point $x_0 \neq 0$ follows by Theorem 4.2. Therefore, we only need to show that if u vanishes of infinite order at $x = 0$ then $u \equiv 0$ in B_1. We set for $r \in (0,1)$

(4.27) $$H(r) = \int_{\partial B_r} u^2 dH_{n-1}, \qquad I(r) = \int_{B_r} \left(|\nabla u|^2 + \frac{f(\omega)}{|x|^2} u^2 \right) dx.$$

A computation similar to those in (4.8) and (4.9) gives

(4.28) $$I'(r) = \frac{n-2}{r} \int_{B_r} |\nabla u|^2 dx + 2 \int_{\partial B_r} u_\rho^2 dH_{n-1} + \int_{\partial B_r} \frac{f(\omega)}{|x|^2} u^2 dH_{n-1} - \frac{1}{r} \int_{B_r} \langle x, \nabla(u^2) \rangle \frac{f(\omega)}{|x|^2} dx.$$

Integrating by parts in the last term on the right-hand side of (4.28), recalling that $\partial f/\partial r = 0$, yields

(4.29) $$-\sum_{i=1}^n \frac{1}{r} \int_{B_r} x_i (u^2)_{x_i} \frac{f(\omega)}{|x|^2} dx = -\sum_{i=1}^n \frac{1}{r^2} \int_{\partial B_r} u^2 f(\omega) \frac{x_i^2}{|x|^2} dH_{n-1} + \sum_{i=1}^n \frac{1}{r} \int_{B_r} u^2 \left(f(\omega) \frac{x_i}{|x|^2} \right)_{x_i} dx.$$

Now since $(x_i/|x|^2)_{x_i} = (n-2)/|x|^2$ we get from (4.28) and (4.29)

(4.30) $$I'(r) = \frac{n-2}{r} I(r) + 2 \int_{\partial B_r} u_\rho^2 dH_{n-1}.$$

Formula (4.30) is remarkably simple if compared, for example, to (4.9). If we set $N(r) = rI(r)/H(r)$, and we use (4.6), we obtain

$$N'(r) = N(r)\left\{ 2\frac{\int_{\partial B_r} u_\rho^2}{\int_{\partial B_r} uu_\rho} - 2\frac{\int_{\partial B_r} uu_\rho}{\int_{\partial B_r} u^2} \right\}. \tag{4.31}$$

Since $N(r)$ always has the same sign as $I(r) = \int_{\partial B_r} uu_\rho dH_{n-1}$, from (4.31) and Schwarz' inequality we have

$$N'(r) \geq 0 \qquad \text{for a.e. } r \in (0,1). \tag{4.32}$$

Therefore $N(r)$ is an increasing function of r and from this we conclude as in Theorem 1.3 that (3.5) holds for every couple of concentric balls, B_R and B_{2R}, centered at $x = 0$ and such that $2R < 1$. This immediately implies strong unique continuation.

We close the proof of the theorem with a remark. Not obvious a priori, a by-product of the monotonicity (4.32) is that

$$H(r) \neq 0 \qquad \text{for every } r \in (0,1). \tag{4.33}$$

To see this, we observe that (4.30) implies that

$$\frac{I(r)}{r^{n-2}} \text{ is monotone increasing on } (0,1). \tag{4.34}$$

Therefore, since $u \not\equiv 0$, there exists at most one $r_0 \in (0,1)$ such that $H(r_0) = 0$. In fact, if $r_1 \in (0,1)$ were another value for which $H(r_1) = 0$, we would have $u_{|\partial B_{r_0}} = u_{|\partial B_{r_1}} = 0$ hence $I(r_0) = I(r_1) = 0$, which contradicts $u \not\equiv 0$. By (4.6) we get

$$\left(\log \frac{H(r)}{r^{n-1}}\right)' < \frac{2N(1)}{r}, \tag{4.35}$$

where we have set $N(1) = \sup_{(0,1)} N(r)$. Integrating (4.35) gives

$$\int_{r_0}^{r} \left(\log \frac{H(t)}{t^{n-1}}\right)' dt \leq K$$

hence

$$\log \frac{H(r)}{r^{n-1}} - \log \frac{H(r_0^+)}{r_0^{n-1}} \leq K, \tag{4.36}$$

where $H(r_0^+) = \lim_{r \to r_0^+} H(r)$. (4.36) implies $H(r_0^+) \neq 0$. This in turn gives that (4.32) actually holds everywhere on $(0,1)$.

We wish to end this section with

A counterexample. Our point here is to show that the inverse square potential $V(x) = c/|x|^2$, $c \in \mathbf{R}$, plays the role of a threshold, among all potentials of the type $c/|x|^m$, $m \geq 0$, for unique continuation to hold. In fact, one can use the same ideas of Theorems 4.1 and 4.2 to show unique continuation for $-\Delta + c|x|^{-m}$, when $0 < m < 2$. Note that for $0 < m < 2$, $|x|^{-m} \in L^p_{\text{loc}}$ for some $p > n/2$, therefore, see also [JK]. We now consider the Schrödinger equation

$$(4.37) \qquad Hu = -\Delta u + \frac{c}{|x|^{2+\varepsilon}} u = 0 \qquad \text{in } B_1,$$

and we suppose that the dimension n of the ambient space is such that

$$(4.38) \qquad \frac{n-2}{\varepsilon} \notin \mathbf{Z}^+.$$

If we look for radial soutions $u(x) = u(|x|)$ of (4.37) we are led to consider the ode

$$(4.39) \qquad r^2u''(r) + (n-1)ru'(r) - cr^{-\varepsilon}u(r) = 0, \qquad 0 < r < 1.$$

(4.39) is a Bessel equation. If we have the general equation

$$(4.40) \qquad z^2u''(z) + (1-2\alpha)zu'(z) + [\beta^2\gamma^2z^{2\gamma} + (\alpha^2 - v^2\gamma^2)]u(z) = 0,$$

then for $v \notin \mathbf{Z}$ the general solution of (4.40) is given by

$$(4.41) \qquad u(z) = z^\alpha[C_1J_v(\beta z^\gamma) + C_2J_{-v}(\beta z^\gamma)],$$

C_1, C_2 being arbitrary complex numbers, see [W]. Comparison of (4.39) with (4.40) gives

$$(4.42) \qquad \alpha = -\frac{n-2}{2}, \qquad \gamma = -\varepsilon/2, \qquad v = \frac{n-2}{\varepsilon},$$

and, if we take $c > 0$ in (4.37),

$$(4.43) \qquad \beta = i\frac{2\sqrt{c}}{\varepsilon}.$$

Because of (4.38) the general solution of (4.39) is therefore given by

$$(4.44) \quad u(x) = |x|^{-(n-2)/2}\left[C_1J_v\left(e^{i\pi/2}\frac{2\sqrt{c}}{\varepsilon}|x|^{-\varepsilon/2}\right) + C_2\,J_{-v}\left(e^{i\pi/2}\frac{2\sqrt{c}}{\varepsilon}|x|^{-\varepsilon/2}\right)\right],$$

$C_1, C_2 \in \mathbf{C}$ arbitrary. If now I_v denotes the modified Bessel function of imaginary argument by the formula (see [W])

$$(4.45) \qquad J_v(e^{i\pi/2}z) = e^{i\pi v/2}I_v(z), \qquad -\pi < \arg z < \pi/2,$$

and (4.44), we get

$$(4.46)\quad u(x) = |x|^{-(n-2)/2}\left[C_1 \exp\left[\frac{i\pi}{2}\left(\frac{n-2}{\varepsilon}\right)\right] I_{(n-2)/\varepsilon}\left(\frac{2\sqrt{c}}{\varepsilon}|x|^{-\varepsilon/2}\right)\right.$$

$$\left. + C_2 \exp\left[\frac{i\pi}{2}\left(\frac{n-2}{\varepsilon}\right)\right] I_{-(n-2)/\varepsilon}\left(\frac{2\sqrt{c}}{\varepsilon}|x|^{-\varepsilon/2}\right)\right]$$

Denoting by K_v the modified Bessel function of the third kind, using the formula ([W])

$$K_v(z) = \frac{\pi}{2}\frac{e^{-i\pi v/2}I_{-v}(z) - e^{-i\pi v/2}I_v(z)}{\sin \pi v}, \qquad -\pi < \arg z < \pi/2,$$

and choosing C_1, C_2 in (4.46) such that

$$C_1 \exp\left[\frac{i\pi}{2}\left(\frac{n-2}{\varepsilon}\right)\right] = \frac{\pi}{2}\frac{-\exp\left[-\frac{i\pi}{2}\left(\frac{n-2}{\varepsilon}\right)\right]}{\sin \pi\left(\frac{n-2}{\varepsilon}\right)},$$

$$C_2 \exp\left[\frac{i\pi}{2}\left(\frac{n-2}{\varepsilon}\right)\right] = \frac{\pi}{2}\frac{\exp\left[-\frac{i\pi}{2}\left(\frac{n-2}{\varepsilon}\right)\right]}{\sin \pi\left(\frac{n-2}{\varepsilon}\right)},$$

we obtain from (4.46)

$$(4.47)\qquad u(x) = |x|^{-(n-2)/2}K_{(n-2)/\varepsilon}\left(\frac{2\sqrt{c}}{\varepsilon}|x|^{-\varepsilon/2}\right).$$

Now we use the asymptotic behavior of K_v for large positive x, i.e.,

$$(4.48)\qquad K_v(x) \approx \frac{\pi}{2}x^{-1/2}e^{-x} \qquad \text{as } x \to +\infty.$$

By (4.47) it is then clear that

$$u(x) = |x|^{-(n-2)/2}K_{(n-2)/\varepsilon}\left(\frac{2\sqrt{c}}{\varepsilon}|x|^{-\varepsilon/2}\right)$$

is a nontrivial solution of (4.37) which, by (4.48), vanishes of infinite order at $x = 0$.

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