Operator Theory:
Advances and Applications, Vol. 261, 333–344

Nodal Sets of Laplace Eigenfunctions: Estimates of the Hausdorff Measure in Dimensions Two and Three

Alexander Logunov and Eugenia Malinnikova

In memory of our teacher Victor Petrovich Havin

Abstract. Let Δ_M be the Laplace operator on a compact n-dimensional Riemannian manifold without boundary. We study the zero sets of its eigenfunctions $u : \Delta_M u + \lambda u = 0$. In dimension $n = 2$ we refine the Donnelly–Fefferman estimate by showing that $\mathcal{H}^1(\{u = 0\}) \le C\lambda^{3/4-\beta}$ for some $\beta \in (0, 1/4)$. The proof employs the Donnelly–Fefferman estimate and a combinatorial argument, which also gives a lower (non-sharp) bound in dimension $n = 3$: $\mathcal{H}^2(\{u = 0\}) \ge c\lambda^{\alpha}$ for some $\alpha \in (0, 1/2)$. The positive constants c, C depend on the manifold, α and β are universal.

Mathematics Subject Classification (2010). Primary 31B05. Secondary 35R01, 58G25.

Keywords. Laplace eigenfunctions, nodal set, harmonic functions.

1. Introduction

Let Δ_M be the Laplace operator on a compact n-dimensional Riemannian manifold without boundary. It was conjectured by Yau, see [18], that the nodal sets $E_\lambda = \{u_\lambda = 0\}$ of Laplace eigenfunctions u_λ, $\Delta_M u_\lambda + \lambda u_\lambda = 0$ satisfy the following inequality

$$C_1\sqrt{\lambda} \le \mathcal{H}^{n-1}(E_\lambda) \le C_2\sqrt{\lambda}.$$

This conjecture was proved by Donnelly and Fefferman under the assumption that the Riemannian metric is real-analytic ([3]). The left-hand side estimate was also proved for smooth non-analytic surfaces by Brüning ([1]).

A. L. was supported in part by ERC Advanced Grant 692616 and ISF Grants 1380/13, 382/15. Eu. M. was supported by Project 213638 of the Research Council of Norway.

The previous best known estimate from below for a non-analytic manifold in higher dimensions is

$$\mathcal{H}^{n-1}(E_\lambda) \geq C\lambda^{(3-n)/4},$$

which gives a constant for $n = 3$. The two known approaches are: (1) follow the ideas of Donnelly and Fefferman and find many balls on the wave-scale $\lambda^{-1/2}$ with bounded doubling index, as it is done in [2] or (2) use the Green formula $2\int_{E_\lambda} |\nabla_M u_\lambda| = \lambda \int_M |u_\lambda|$ and the estimate $\frac{\|u_\lambda\|_\infty}{\|u_\lambda\|_1} \leq C\lambda^{(n-1)/4}$, see [16]. The approach in [2] also exploits the Sogge–Zelditch estimates of L^p-norms of eigenfunctions. The following upper estimate in dimension two was established by Donnelly and Fefferman, see [4],

$$\mathcal{H}^1(E_\lambda) \leq C\lambda^{3/4}.$$

In this paper we obtain tiny improvements to the estimate from below in dimension three and to the estimate from above in dimension two. We show that in dimension 2

$$\mathcal{H}^1(E_\lambda) \leq C\lambda^{3/4-\beta}, \tag{1.1}$$

for some $\beta \in (0, 1/4)$. It gives a small refinement to the Donnelly–Fefferman estimate. The proof of (1.1) relies on the results and methods from [3, 4]. Roughly speaking, the Donnelly–Fefferman argument, which gives the estimate with $\frac{3}{4}$, is combined with a combinatorial argument presented below, which gives the β improvement. The same combinatorial argument shows that in dimension $n = 3$

$$\mathcal{H}^2(E_\lambda) \geq C\lambda^\alpha, \tag{1.2}$$

for some $\alpha > 0$. As far is we know it gives the first bound that grows to infinity as λ increases, but we note that the last result is not sharp and can be improved up to the bound $c\sqrt{\lambda} \leq \mathcal{H}^{n-1}(E_\lambda)$ conjectured by Yau.

This paper is the first part of the work, which consists of three parts. Polynomial upper estimates for the Hausdorff measure of the nodal sets in higher dimensions are proved in the second part [9] by a new technique of propagation of smallness. The lower bound in Yau's conjecture is proved in the third part [10] as well as its harmonic counterpart (Nadirashvili's conjecture). We remark that the results in [9, 10] do not give the estimate (1.1) and all three parts can be read independently.

2. Toolbox

2.1. Inequalities for solutions of elliptic equations

Let $(\mathcal{M}, g)$ be a smooth Riemannian manifold and $\Delta_{\mathcal{M}}$ the Laplace operator on $\mathcal{M}$, which is defined by the metric g. We always assume that the metric is fixed. In the sequel we consider $\mathcal{M} = M \times \mathbb{R}$, where M is a compact manifold with a given metric, on which we study the eigenfunctions, and $\mathcal{M}$ is endowed with the usual metric of the product. Although $\mathcal{M}$ is not compact itself, we will always work on the compact subset $P = M \times [-1, 1]$ of $\mathcal{M}$ where all our estimates are uniform.

A function h on $\mathcal{M}$ is called harmonic if it satisfies the elliptic equation

$$L(h) = \operatorname{div}(\sqrt{g}(g^{ij})\nabla h) = 0 \tag{2.1}$$

in local coordinates. More precisely, the Laplace operator on $\mathcal{M}$ is given by $\Delta_{\mathcal{M}}(f) = \frac{1}{\sqrt{g}} \operatorname{div}(\sqrt{g}(g^{ij})\nabla f)$. Harmonic functions satisfy the maximum and minimum principles and the standard elliptic gradient estimates, see for example [5, Chapter 3]. Further, there exists a constant C such that for any geodesic ball $B(x,r) \subset P$

$$|\nabla h(x)| \leq \frac{C}{r} \sup_{B(x,r)} |h|. \tag{2.2}$$

The Harnack inequality holds: if h satisfies (2.1) and $h > 0$ in $B(x,r)$, then for any $y \in B(x, \frac{2}{3}r)$

$$\frac{1}{C} h(y) < h(x) < C h(y). \tag{2.3}$$

The following consequence of the Harnack inequality will be also used later. If h satisfies (2.1) and $h(x) \geq 0$ then

$$\sup_{B(x,r)} h \geq c \sup_{B(x,\frac{2}{3}r)} |h| \tag{2.4}$$

for some $c = c(\mathcal{M}) > 0$ (it follows from the Harnack inequality applied to the function $\sup_{B(x,r)} h - h$).

2.2. Estimates on the wavelength scale

Now let (M, g_0) be a compact Riemannian manifold. We consider a Laplace eigenfunction u which satisfies $\Delta_M u = -\lambda u$. Adding a new variable, we consider the function $h(\xi, t) = u(\xi) e^{\sqrt{\lambda} t}$ on the product manifold $\mathcal{M} = M \times \mathbb{R}$. The function h turns out to be harmonic on $\mathcal{M}$. This observation can be used to claim that on the wave-scale $\lambda^{-1/2}$ the behavior of the Laplace eigenfunctions resembles that of harmonic functions. This well-known trick was successfully exploited for example in [8, 14, 11].

Let ξ be an arbitrary point on M. Denote by $B(\xi, r) = B_r(\xi)$ the geodesic ball with center at ξ of radius r, when the center of the ball is not important we will omit it in the notation and write B_r.

Lemma 2.1. *There exist a small number $\varepsilon = \varepsilon(M) > 0$ and constants $C_1 = C_1(M)$, $C_2 = C_2(M)$ such that for any eigenfunction u, $\Delta_M u = -\lambda u$, and any $r < \varepsilon \lambda^{-1/2}$ the following inequalities hold*

$$\text{(a)} \sup_{B_r} |u| \leq 2 \max_{\partial B_r} |u|, \quad \text{(b)} \sup_{B_{\frac{1}{2}r}} |\nabla u| \leq C_1 \max_{\partial B_r} |u|/r. \tag{2.5}$$

If, in addition, $u(\xi) \geq 0$ and (therefore) $A = \max_{\partial B_r(\xi)} u \geq 0$, then

$$\sup_{B_{\frac{2}{3}r}(\xi)} |u| \leq C_2 A. \tag{2.6}$$

The inequalities (2.5) and (2.6) follow from the standard elliptic estimates, we provide the proofs for the convenience of the reader.

We work in local coordinates on $\mathcal{M} = M \times \mathbb{R}$ and consider the harmonic function $h(\eta, t) = u(\eta)e^{\sqrt{\lambda}t}$ on $\mathcal{M}$. The Laplace operator corresponds (locally) to an elliptic operator L defined on a bounded subdomain Ω of $\mathbb{R}^{n+1}$, see above. We choose local coordinates such that the distance on the manifold is equivalent to the Euclidean distance (for example by choosing normal coordinates). Denote by $G_{\Omega,L}$ the Green function for L on Ω. By $|x-y|$ we denote the ordinary Euclidean distance between points x and y, locally $|x-y|$ is comparable to the distance between the corresponding points in the Riemannian metric on $\mathcal{M}$. We use the following upper estimate (see [17], [15], [7]) of the Green function:

$$G_{\Omega,L}(x,y) \leq \frac{C}{|x-y|^{d-2}},$$

where $d = n+1$ is the dimension of $\mathcal{M}$. The constant C depends on the coordinate chart on $\mathcal{M}$, we consider a finite set of charts that covers $M \times [-1,1]$.

Proof of Lemma 2.1. First we suppose that $\sup_{B_r(\xi)} u > 0$ and prove that

$$\sup_{B_r(\xi)} u \leq 2 \max_{\partial B_r(\xi)} u, \tag{2.7}$$

if $r < \varepsilon(M)\lambda^{-1/2}$, where $\varepsilon(M)$ is a sufficiently small positive number, which will be chosen later. Put $A = \max_{\partial B_r(\xi)} u$ and $K = \sup_{B_r(\xi)} u$. Let ξ_0 be a point in the closed ball $\overline{B_r}$, where K is attained.

We consider the cylinder $Q = B_r \times (\frac{-1}{2\sqrt{\lambda}}, \frac{1}{2\sqrt{\lambda}})$ on $\mathcal{M}$. We have $\sup_Q h \leq \sqrt{e}K$ on Q. Without loss of generality we assume that $Q \subset \Omega$. First, we prove $A > 0$.

Suppose that $A \leq 0$ and define the function $w(\xi,t) := A(1-4\lambda t^2)/\sqrt{e} + 4\sqrt{e}K\lambda t^2$ and note that $w \geq h$ on ∂Q and $|Lw| < C_3(K-A)\lambda$ on Q for some $C_3 = C_3(M)$. Now, consider the difference $v = h - w$. It is non-positive on ∂Q and satisfies $v(\xi_0, 0) = K - A/\sqrt{e}$ and $|Lv| \leq C_3(K-A)\lambda$ on Q.

We can decompose v into the sum $v = g_1 + g_2$, where g_1 is a non-positive harmonic function in Q with $g_1|_{\partial Q} = v|_{\partial Q}$ and $g_2(y) = \int_Q G_{Q,L}(x,y)Lv(x)dx$. Since $Q \subset \Omega$, the Green function satisfies

$$G_{Q,L}(x,y) \leq G_{\Omega,L}(x,y) \leq C_4/|x-y|^{d-2}.$$

Further, for any $y \in Q$ a simple estimate gives

$$\int_Q |x-y|^{2-d}dx \leq C_5 \frac{r}{\sqrt{\lambda}} \leq C_5 \frac{\varepsilon}{\lambda}.$$

Combining the estimates, we get

$$g_2(y) = \int_Q G_{Q,L}(x,y)Lv(x)dx \leq C_3C_4(K-A)\lambda \int_Q |x-y|^{2-d}dx \leq C_6(K-A)\varepsilon.$$

Hence $g_2(\xi_0, 0) \le C_6(K - A)\varepsilon$. The function g_1 is non-positive in Q and therefore

$$K - A/\sqrt{e} = v(\xi_0, 0) = g_1(\xi_0, 0) + g_2(\xi_0, 0) \le C_6(K - A)\varepsilon.$$

Thus $A(1/\sqrt{e} - C_6\varepsilon) \ge K(1 - C_6\varepsilon)$ and if ε is sufficiently small we obtain that $A > 0$.

Now, we know that $A \ge 0$ and we repeat the argument above with $\tilde{w}(\xi, t) = \sqrt{e}A + 4\sqrt{e}K\lambda t^2$ in place of w and obtain

$$K - \sqrt{e}A \le C_6 K\varepsilon.$$

If ε is chosen sufficiently small, then (2.7) follows. Inequality (2.5 (a)) follows from (2.7) if one replaces u by $-u$. Finally, the inequalities (2.5 (b)) and (2.6) are obtained by combining (2.5 (a)) with (2.2) and (2.4) respectively, where the last two inequalities are applied to the harmonic function $h(\xi, t) = u(\xi)e^{\sqrt{\lambda}t}$. □

2.3. Doubling index

Let h be a harmonic function on $\mathcal{M}$. Locally h can be regarded as a solution to the elliptic equation $Lh = 0$. We identify h with a function on the cube $\mathcal{K}_\rho^d = [-\rho, \rho]^d \subset \mathbb{R}^d, d = n + 1$. We choose local geodesic coordinates, then the metric is locally equivalent to the Euclidean one and L is a small perturbation of the Euclidean Laplace operator. Let l be a positive odd integer such that $l > 2\sqrt{d}$, $l = 2l_0 + 1$. For each cube q in $\mathcal{K}_\rho^d$ let lq denote the cube obtained from q by the homothety with the center at the center of q and coefficient l. Suppose that $lq \subset \mathcal{K}_\rho^d$, then we define the doubling index $N(h, q)$ by

$$\int_{lq} |h(x)|^2 dx = 2^{N(h,q)} \int_q |h(x)|^2 dx.$$

The notion of doubling index was used for estimates of the nodal sets in [3, 4, 8] and in many subsequent works. We will need the following properties of the doubling index.

Lemma 2.2.

(i) (L^∞*-estimate*) *If a cube* q *is inscribed in a ball* B (*and therefore* lq *contains* $2B$), *then*

$$\sup_{\frac{4}{3}B} |h| \le C_7 2^{N(h,q)/2} \sup_B |h|,$$

for some positive $C_7 = C_7(M)$.

(ii) (*Monotonicity property*) *There exists a positive integer* $A = A(d)$, *a constant* $C_0 = C_0(d) > 1$ *and a positive number* $\rho = \rho(M)$ *such that if* q_1 *and* q *are cubes that are contained in* $\mathcal{K}_\rho^d$, *and* $Aq_1 \subset q$ *then* $N(h, q_1) \le C_0 N(h, q)$.

Proof. (i) Indeed, we have

$$\int_{2B} |h|^2 \le \int_{lq} |h|^2 = 2^{N(h,q)} \int_q |h|^2 \le 2^{N(h,q)} \int_B |h|^2.$$

Clearly, $\int_B |h|^2 \le (\sup_B |h|)^2 |B|$. Further, by an elliptic estimate for h, $\sup_{\frac{4}{3}B} |h| \le C \left(\int_{2B} |h|^2\right)^{1/2} |B|^{-1/2}$. The inequality follows.

(ii) The monotonicity property is left without proof. We refer to [6] and [11] for the proof of the monotonicity property of the doubling index defined through integrals over concentric geodesic spheres instead of cubes. Using this, it is not difficult to derive the monotonicity property of doubling index for cubes instead of spheres. □

3. Inscribed balls and a local estimate of the volume of the nodal set

The aim of this section is to estimate from below the volume of the nodal set of an eigenfunction u of the Laplace operator in a geodesic ball of radius comparable to the wavelength $\lambda^{-1/2}$, where $\Delta_M u + \lambda u = 0$. The estimates presented in this section are very far from being sharp.

Let us fix a point O on M and assume $u(O) = 0$. Denote by $|x|$ the distance from the point x to O. We will consider the geodesic ball B_r of radius $r \le \varepsilon\lambda^{-1/2}$ and with center at O, where $\varepsilon = \varepsilon(M)$ is chosen so that the inequalities (2.5) and (2.6) hold.

Lemma 3.1. *Assume that* $\sup_{B_{\frac{r}{2}}} |u| \le 2^N \sup_{B_{\frac{r}{4}}} |u|$, *where* N *is a positive integer,* $N \ge 4$. *Then*

$$\mathcal{H}^{n-1}\{|x| \le r/2, u(x) = 0\} \ge cr^{n-1}N^{2-n}, \tag{3.1}$$

for some positive $c = c(M)$.

Proof. Applying (2.6), one can deduce

$$\frac{\max_{\partial B_{r/2}} u}{\max_{\partial B_{3r/8}} u} \le C_2 \frac{\sup_{B_{r/2}} |u|}{\sup_{B_{r/4}} |u|} \le C_2 2^N.$$

Let $S_j = \{x : |x| = r_j = r(\frac{3}{8} + \frac{j}{8N})\}$, $m_j^+ = \max_{S_j} u$ and $m_j^- = \min_{S_j} u$, $j = 0, 1, \dots, N$. Recall that u is zero at O. It follows from the weak maximum principle (2.7) that

$$m_j^- < 0,\ m_j^+ > 0 \quad \text{and} \quad m_j^+ \le 2m_{j+1}^+,\ |m_j^-| \le 2|m_{j+1}^-|.$$

We consider the ratios $\tau_j = m_{j+1}^+/m_j^+$, $j = 0, \dots, N-1$. Then each $\tau_j \ge 1/2$ and

$$\tau_0 \dots \tau_{N-1} = \frac{\max_{\partial B_{r/2}} u}{\max_{\partial B_{3r/8}} u} \le C_2 \frac{\sup_{B_{r/2}} |u|}{\sup_{B_{r/4}} |u|} \le C_2 2^N.$$

Therefore at most $N/4$ of the ratios $\tau_0, \dots, \tau_{N-1}$ are greater than a sufficiently large constant $C_3 = C_3(C_2)$. Similarly, at most $N/4$ of the ratios $|m_{j+1}^-|/|m_j^-|$ are greater than C_3. Hence there are at least $N/2$ numbers k, $0 \le k \le N-1$ such that $m_{k+1}^+ \le C_3 m_k^+$ and $|m_{k+1}^-| \le C_3 |m_k^-|$. We want to show that for each such k there is a ball of radius cr/N and centered on the sphere S_k where u is positive.

Indeed, let x_0 be such that $|x_0| = r_k$ and $u(x_0) = m_k^+ = \max_{\{|x|=r_k\}} u(x)$ and let b be the ball centered at x_0 with radius $\frac{r}{16N}$. Then

$$\sup_b u \le \max_{\{|x|\le r_{k+1}\}} u(x) \le C_1 \max_{\{|x|=r_{k+1}\}} u(x) \le C_4 m_k^+.$$

Applying (2.6) we see that $\max_{\frac{1}{2}b} |u| \le C_5 m_k^+$. Taking into account (2.5) (b) and $u(x_0) = m_k^+$, we deduce that u is positive in a smaller ball of radius $c_1 r/N$ centered at x_0.

Similarly, we can find a ball of radius $c_1 r/N$ with center on S_k where u is negative. Thus the spherical layer $\{x : r_{k-1} < |x| < r_{k+1}\}$ contains two balls of radius $c_1 r/N$ where u has opposite signs. Then

$$\mathcal{H}^{n-1}\{x : r_{k-1} < |x| < r_{k+1} : u(x) = 0\} \ge c_2 \left(\frac{r}{N}\right)^{n-1}.$$

The last inequality holds for at least $N/2$ numbers k, so (3.1) follows. □

4. Combinatorial argument

We need the following lemma about the doubling index defined in Section 2.3. This lemma holds for an arbitrary function $h \in L^2$, not necessarily harmonic.

Lemma 4.1. *Let a cube Q be partitioned into $(Kl)^d$ equal cubes q_i with side length $\frac{1}{Kl}$ (where l is the odd integer from the definition of the doubling index and K is an arbitrary positive integer). Put $N_{\min} = \min_i N(h, q_i)$, the minimum is taken over those cubes q_i of the partition for which $lq_i \subset Q$, and assume that $N_{\min} \ge 2d \ln l / \ln 2$. Then $N(h, \frac{1}{l}Q) \ge \frac{1}{2} K N_{\min}$.*

Proof. Define $Q_j = \frac{K + j(l-1)}{Kl} Q$ for $j = 0, 1, \dots, K$, in particular $Q_0 = \frac{1}{l}Q$ and $Q_K = Q$.

We know that $\int_{lq_i} |h|^2 \ge 2^{N_{\min}} \int_{q_i} |h|^2$ for each q_i and therefore

$$2^{N_{\min}} \int_{Q_j} |h|^2 \le \sum_{q_i \subset Q_j} \int_{lq_i} |h|^2 \le l^d \int_{Q_{j+1}} |h|^2,$$

since the union of the (open) cubes $lq_i, q_i \subset Q_j$, is contained in Q_{j+1} and covers each point of Q_{j+1} with multiplicity at most l^d.

Further, the inequality $N_{\min} \ge 2d \ln l / \ln 2$ implies

$$2^{N_{\min}/2} \int_{Q_j} |h|^2 \le \int_{Q_{j+1}} |h|^2.$$

Finally, multiplying the last inequalities for $j = 0, \dots, K-1$, we obtain

$$\int_{Q_K} |h|^2 \ge 2^{K N_{\min}/2} \int_{Q_0} |h|^2 = 2^{K N_{\min}/2} \int_{\frac{1}{l}Q} |h|^2.$$

□

Suppose now that h is a harmonic function on $\mathcal{M} = M \times \mathbb{R}$. Given a cube c, define $\tilde{N}(h,c) = \sup_{c' \subset c} N(h,c')$, where the supremum is taken over all subcubes c' of the cube c. The monotonicity property implies

$$\tilde{N}\left(h, \frac{1}{A}c\right) \le C_0 N(h,c),$$

when c is contained in $\mathcal{K}_\rho^{d+1}$ and ρ is sufficiently small. If a cube c contains a cube c', then $\tilde{N}(h,c) \ge \tilde{N}(h,c')$.

Our aim is to divide the cube $q = \mathcal{K}_{\rho/l}^{d+1}$ into small cubes and estimate the number of cubes with large doubling constants.

Lemma 4.2. *Let h be a solution to $Lh = 0$ in q. There exist constants $B_0 = B_0(d,L)$ and $\delta = \delta(d) > 0$ such that if the cube q is partitioned into $B > B_0$ equal subcubes, then at least half of these subcubes c satisfy*

$$\tilde{N}(h,c) \le \max\left\{ \frac{\tilde{N}(h,q)}{B^\delta}, \frac{2d \ln l}{\ln 2} \right\}.$$

Proof. Let $N_0 = \tilde{N}(h,q)$. We will do the partition step by step. At the beginning we have one cube q with $\tilde{N}(h,q) = N_0$. We fix A and C_0 from the monotonicity property of the doubling index and choose an integer K such that $K > 4C_0$.

On the first step we divide q into $Y = [lKA]^d$ subcubes. First, divide q into $[lK]^d$ subcubes. By Lemma 4.1 at least one subcube c satisfies $N(h,c) \le 2N_0/K$ if N_0 is sufficiently large. Then $\tilde{N}(h, \frac{1}{A}c) \le 2C_0 N_0/K \le N_0/2$. Thus if we divide q into $[lKA]^d$ subcubes, then at least one subcube will have $\tilde{N} \le N_0/2$ and all other subcubes will have $\tilde{N} \le N_0$.

On the second step we will repeat the partition procedure in each subcube c from the first step. Then at least one subcube c' of c will have $\tilde{N}(h,c') \le \tilde{N}(h,c)/2$. Also $\tilde{N}(h,c'') \le \tilde{N}(h,c)$ for any other subcube c'' of c.

Going from the $(j-1)$st to the jth step, we take any cube c from the previous step and divide it into Y equal subcubes. In each cube with $\tilde{N}(h,c) \le N_0/2^s$, $1 \le s \le j$, we get at least one cube c' with $\tilde{N}(h,c') \le N_0/2^{s+1}$ and for other cubes in c we have $\tilde{N}(h,c') \le N_0/2^s$.

Using the standard induction argument, one can see that on the jth step there is one cube with the doubling index less than or equal to $N_0/2^j$, $\binom{j}{1}(Y-1)$ other cubes with the indices less than or equal to $N_0/2^{j-1}$, and so on, with $\binom{j}{k}(Y-1)^{j-k}$ other cubes with the indices smaller than $N_0/2^k$, $k \ge 0$ (assuming that $N_0/2^j \ge 2d \ln l/\ln 2$). The sum $\sum_{k=0}^{j} \binom{j}{k}(Y-1)^{j-k} = (1 + (Y-1))^j$ is the number of all cubes on the jth step.

Let $\xi_1, \ldots, \xi_j$ be i.i.d. random variables such that $\mathcal{P}(\xi_1 = 1) = 1/Y$ and $\mathcal{P}(\xi_1 = 0) = (Y-1)/Y$. By the law of large numbers

$$\mathcal{P}\left(\frac{\sum_{i=1}^{j} \xi_i}{j} > \frac{1}{2Y}\right) \to 1 \quad \text{as } j \to \infty.$$

If j is sufficiently large, then

$$\frac{1}{2} \leq \mathcal{P}\left(\sum_{i=1}^{j} \xi_i \geq \frac{j}{2Y}\right) = \sum_{j \geq k \geq \frac{j}{2Y}} \mathcal{P}\left(\sum_{i=1}^{j} \xi_i = k\right) = \sum_{k \geq \frac{j}{2Y}} \binom{j}{k} \frac{(Y-1)^{j-k}}{Y^j}.$$

We conclude that at least half of all cubes on the jth step have doubling indices bounded by $N_0/2^{\frac{j}{2Y}}$. Let $B = [lKA]^{jd} = Y^j$, then $N_0/2^{\frac{j}{2Y}} \leq N_0/B^\delta$, where $\delta = \delta(Y)$ is a positive number such that $Y^\delta < 2^{\frac{1}{4Y}}$ and thus δ depends only on the dimension d. Here we have assumed that $j > j_0$ to apply the law of large numbers and we have also assumed that $N_0/B^\delta \geq 2d \ln l / \ln 2$ to apply Lemma 4.1. □

5. Estimates of the nodal sets of eigenfunctions

5.1. Lower estimate in dimension three

Suppose now that u is a Laplace eigenfunction, $\Delta_M u + \lambda u = 0$ on M, where M is a smooth Riemannian three-dimensional manifold. Using the standard trick, we consider the manifold $\mathcal{M} = M \times \mathbb{R}$ and a new function $h(\xi, t) = u(\xi)e^{\sqrt{\lambda}t}$, which satisfies $\Delta_{\mathcal{M}} h = 0$. We therefore work on a four-dimensional manifold.

We fix a cube Q on M and consider the cube $\tilde{Q} = Q \times I$ on $\mathcal{M}$, where I is the interval centered at the origin with the length equal to the side length of Q, we choose Q sufficiently small such that a chart for Q in normal coordinates is contained in some $\mathcal{K}_\rho^4$.

The Donnelly–Fefferman estimate, see [3], implies that $\tilde{N}(u, Q) \leq C\sqrt{\lambda}$ for some $C = C(M)$ if the diameter of Q is less than $c(M)$, and therefore $\tilde{N}(h, \tilde{Q}) \leq C_1\sqrt{\lambda}$. See also [11] for the explanation of the Donnelly–Fefferman estimate via the three sphere theorem for harmonic functions.

We partition $\tilde{Q}$ into B smaller cubes $\tilde{q}$ with the side length of order $\lambda^{-1/2}$, such that for each small cube $\tilde{q}$ there is a zero of h within $\frac{1}{10}\tilde{q}$ (it is well known, see [3], that the nodal set of u is $c\lambda^{-1/2}$ dense on M). Then $B \sim [c\sqrt{\lambda}]^4 \sim c_1\lambda^2$ and B is large enough when $\lambda > \lambda_0$.

By Lemma 4.2, half of all small cubes have doubling indices bounded by $C\sqrt{\lambda}/B^\delta \leq C_1\lambda^{1/2-2\delta}$. In each small cube of the wavelength size $C/\sqrt{\lambda}$ the doubling index for h is comparable to the doubling index for the function u on the projection of the cube to M, since $h(\xi, t) = u(\xi)e^{\sqrt{\lambda}t}$. Then at least one half of the small cubes of size $C/\sqrt{\lambda}$ in Q have doubling indices bounded by $C_2\lambda^{1/2-2\delta}$. In each such cube q we can find a smaller subcube q' with diameter $\frac{\varepsilon}{\sqrt{\lambda}}$ such that u is equal to 0 at the center of q'. Then combining Lemma 2.2 (i) and the estimate (3.1), we obtain

$$\mathcal{H}^2(\{u = 0\} \cap q') \geq \frac{c_2}{\lambda N(u, q')} \geq c_3\lambda^{-3/2}\lambda^{2\delta}.$$

The number of such cubes is comparable to $\lambda^{3/2}$. Thus $\mathcal{H}^2(\{u = 0\}) \geq c_4\lambda^{2\delta}$.

5.2. Upper estimate in dimension two

Following [4], using local isothermal coordinates in a geodesic disk of radius r, we transform the eigenfunction u, $\Delta_M u + \lambda u = 0$ to a function f in the unit ball of $\mathbb{R}^2$ that satisfies $\Delta_0 f + \lambda r^2 \psi f = 0$, where Δ_0 is the Euclidian Laplacian and ψ is a bounded function (the bound depends on the metric).

We will combine the combinatorial argument from Section 4 with the following estimate for the length of the nodal set by Donnelly and Fefferman, [4]. Let Q be the unit square.

Suppose that $g : Q \to \mathbb{R}$ satisfies $\tilde{N}(g, Q) \leq \Gamma$, $\Delta g = \Gamma \psi g$, where ψ is a function in Q with sufficiently small L^∞-norm. Then

$$\mathcal{H}^1(x \in \tfrac{1}{100}Q : g(x) = 0) \leq C\Gamma.$$

In [4] this estimate was applied on the scale $\lambda^{-1/4}$: for any square q on M with side $\sim \lambda^{-1/4}$ one can consider a function $u(\lambda^{-1/4}x)$ and apply the estimate with $\Gamma \sim \lambda^{1/2}$ (using that the doubling index for any cube is bounded by $C\lambda^{1/2}$) to see that $H^{n-1}(\{u = 0\} \cap q) \leq C\lambda^{1/4}$. Summing the estimates over such cubes covering M, one has $H^1(\{u = 0\}) \leq C\lambda^{3/4}$.

However a combinatorial argument will show that very few cubes with side $\sim \lambda^{-1/4}$ have doubling indices comparable to $\lambda^{1/2}$, in fact, most of the cubes have significantly smaller doubling indices. We are going to refine the global length estimate via combining the combinatorial argument and the Donnelly–Fefferman estimate on various scales.

Lemma 5.1. *Fix a geodesic ball B on the surface with isothermal coordinates and let q be a square in B with side-length $\sim \lambda^{-1/4}$. Then*

$$\mathcal{H}^1(\{u = 0\} \cap q) \leq C\tilde{N}(u, 100q)^{1/2}. \tag{5.1}$$

Proof. Denote $\tilde{N}(u, 100q)$ by N_0. Let us divide q into squares with side-length $\sim N_0^{1/2}\lambda^{-1/2}$. In each of those the doubling index is bounded by N_0 and rescaling such small squares to unit squares and applying the estimate of Donnelly and Fefferman with $\Gamma = N_0$, we bound the length of the nodal set in such a small square by $CN_0^{3/2}\lambda^{-1/2}$. The number of such squares is $\sim \frac{\lambda^{1/2}}{N_0}$. Then $\mathcal{H}^1(\{u = 0\} \cap q) \leq CN_0^{3/2}\lambda^{-1/2}\lambda^{1/2}N_0^{-1} = CN_0^{1/2}$ □

Recall that $\mathcal{K}_\rho^d = [-\rho, \rho]^d$ and let $\mathcal{K} = \mathcal{K}_\rho^2$ be a square such that $100\mathcal{K}$ lies in (the chart for) B, the side-length of $\mathcal{K}$ depends only on the geometry of the surface M and does not depend on λ. We partition $\mathcal{K}$ into squares with side-length $\lambda^{-1/4}$, then for each such square q we have $\mathcal{H}^1(\{u = 0\} \cap q) \leq C\tilde{N}^{1/2}(u, 100q_0)$ and summing up over all squares q in the partition of $\mathcal{K}$, we obtain

$$\mathcal{H}^1(\{u = 0\} \cap \mathcal{K}) \leq C\sum_{q \subset \mathcal{K}} \tilde{N}^{1/2}(u, 100q).$$

Further, we consider the harmonic extension $h(\xi, t) = u(\xi)e^{\sqrt{\lambda}t}$ of u and let $\tilde{\mathcal{K}} = \mathcal{K}_\rho^3 = \mathcal{K} \times [-\rho, \rho]$. Note that $N(h, \tilde{q}) \geq N(u, q)$, whenever q is the projection of $\tilde{q}$ to M and then the same inequality holds for $\tilde{N}$.

Let Y be a sufficiently large integer defined in Section 4. Choose an integer j such that $Y^j \sim \lambda^{3/4}$. We partition $\tilde{\mathcal{K}}$ into $Y^j \sim \lambda^{3/4}$ subcubes with side-length $\sim \lambda^{-1/4}$. According to Section 4 these cubes can be divided into j groups $G_0, \dots, G_j$ such that $\tilde{N}(h, \tilde{q}) \leq N_0 2^{s-j}$ for each cube $\tilde{q} \in G_s$, where $N_0 := \tilde{N}(h, K_0) \leq C\sqrt{\lambda}$ and the number of cubes in G_s is $\binom{j}{s}(Y-1)^s$. However we need to replace $\tilde{N}(h, \tilde{q})$ by $\tilde{N}(h, 100\tilde{q})$ in the estimate for a number a cubes in order to estimate the sum $\sum_{q \subset \mathcal{K}} \tilde{N}^{1/2}(u, 100q)$. It can be done by changing the parameter l in the definition of the doubling index in Section 2.3. The doubling index with a parameter l in a cube $100c$ can be estimated by the doubling index with a parameter $10000l$ in a cube c. We therefore have $\tilde{N}(h, 100\tilde{q}) \leq CN_0 2^{s-j}$ for each cube $q \in G_s$, here we abuse the notation $\tilde{N}$ for a doubling index with the modified l and denote it by the same letter.

Finally, we apply the inequality $\tilde{N}(u, q) \leq \tilde{N}(h, \tilde{q})$, where q is the projection of $\tilde{q}$, and estimate $\tilde{N}^{1/2}(u, 100q)$ by the average of the corresponding quantities over $Y^{j/3}$ cubes $\tilde{q}$ with the projection q. We obtain

$$\mathcal{H}^1(\{u = 0\} \cap \mathcal{K}) \leq C \sum\nolimits_{q \subset \mathcal{K}} \tilde{N}^{1/2}(u, 100q) \leq CY^{-j/3} \sum\nolimits_{\tilde{q} \subset \tilde{\mathcal{K}}} \tilde{N}^{1/2}(h, 100\tilde{q}).$$

Further we partition all cubes $\tilde{q}$ into the groups G_s,

$$\begin{aligned}\sum_{\tilde{q} \subset \tilde{\mathcal{K}}} \tilde{N}^{1/2}(h, 100\tilde{q}) &= \sum_{s=0}^{j} \sum_{\tilde{q} \in G_s} \tilde{N}^{1/2}(h, 100\tilde{q}) \leq C\lambda^{1/4} \sum_{s=0}^{j} \binom{j}{s} (Y-1)^s 2^{(-1/2)(j-s)} \\ &= C\lambda^{1/4}(Y - 1 + 2^{-1/2})^j.\end{aligned}$$

We have $Y^j = c\lambda^{3/4}$, then $Y - 1 + 2^{-1/2} = Y^{1-\eta}$ for some $\eta = \eta(Y) > 0$ and $\mathcal{H}^1(\{u = 0\}) \leq C_M \lambda^{3/4(1-\eta)}$.

Acknowledgment

Both authors were encouraged to study the topic of nodal geometry by Mikhail Sodin, we are very grateful to Mikhail for his constant support, for the numerous ideas and advice that he was generous to share, and for his comments on the earlier versions of the text. We would also like to thank Lev Buhovsky, who read the draft of this paper and made very helpful suggestions and comments.

This work was started when the first author visited NTNU and the second author visited the Chebyshev Laboratory (SPBSU). The work was finished at TAU and Purdue University. We are grateful to these institutions for their hospitality and for great working conditions.

References

[1] J. Brüning, *Über Knoten von Eigenfunktionen des Laplace–Beltrami Operator*, Math Z. **158** (1978), 15–21.

[2] T.H. Colding, W.P. Minicozzi II, *Lower Bounds for Nodal Sets of Eigenfunctions*, Comm. Math. Phys. **306** (2011), 777–784.

[3] H. Donnelly, C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Invent. Math. **93** (1988), 161–183.

[4] H. Donnelly, C. Fefferman, *Nodal sets for eigenfunctions of the Laplacian on surfaces*, J. Amer. Math. Soc. **3** (1990), 333–353.

[5] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.

[6] N. Garofalo, F.-H. Lin, *Monotonicity properties of variational integrals, A_p-weights and unique continuation*, Indiana Univ. Math. J. **35** (1986), 245–268.

[7] H. Hueber, M. Sieveking, *Continuous bounds for quotients of Green functions*, Arch. Rational Mech. Anal. **89** (1985), no. 1, 57–82.

[8] F.-H. Lin, *Nodal sets of solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math. **44** (1991), 287–308.

[9] A. Logunov, *Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure*, arXiv:1605.02587, to appear in Annals of Math.

[10] A. Logunov, *Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture*, arXiv:1605.02589, to appear in Annals of Math.

[11] D. Mangoubi, *The effect of curvature on convexity properties of harmonic functions and eigenfunctions*, J. Lond. Math. Soc. **87** (2013), 645–662.

[12] D. Mangoubi, *On the inner radius of a nodal domain*, Canad. Math. Bull. **51** (2008), no. 2, 249–260.

[13] D. Mangoubi, *Local asymmetry and the inner radius of nodal domains*, Comm. Partial Differential Equations **33** (2008), no. 9, 1611–1621.

[14] F. Nazarov, L. Polterovich, M. Sodin, *Sign and area in nodal geometry of Laplace eigenfunctions*, Amer. J. Math. **127** (2005), 879–910.

[15] J. Serrin, *On the Harnack inequality for linear elliptic equations*, J. Anal. Math. **4** (1954–1956) no. 1, 292–308.

[16] C.D. Sogge, S. Zelditch, *Lower bounds on the Hausdorff measure of nodal sets II*, Math. Res. Lett. **19** (2012), 1361–1364.

[17] K.-O. Widman, *Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations.* Math. Scand. **21** (1967), 17–37.

[18] S.-T. Yau, *Problem section, Seminar on Differential Geometry*, Annals of Mathematical Studies 102, Princeton, 1982, 669–706.

Alexander Logunov
School of Mathematical Sciences
Tel Aviv University
Tel Aviv 69978, Israel

and

Chebyshev Lab.
St. Petersburg State University
14th Line V.O., 29B
St. Petersburg 199178 Russia
e-mail: log239@yandex.ru

Eugenia Malinnikova
Dept. of Mathematical Sciences
Norwegian University
of Science and Technology
N-7491 Trondheim, Norway

and

Dept. of Mathematics
Purdue University
150 N. University st.
West Lafayette, IN 47907-2067, USA
e-mail:
eugenia.malinnikova@ntnu.no