MATH 125G MIDTERM 1 SOLUTIONS Oct 16, 2008

(1) Evaluate the following indefinite integrals.

(a) (20 points) $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt.$

Solution: Use the substitution $u = \sqrt{t}$ to get $du = \frac{dt}{2\sqrt{t}}\,dt$. We then have

$$\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt \;=\; 2\int \cos u\,du = 2\sin u + C \;=\; 2\sin(\sqrt{t}) \;+\; C\;.$$

(b) (20 points) $\displaystyle\int \frac{\cos x}{\sin^2 x + 1}\,dx.$

Solution: Use the substitution $u = \sin x$ to get $du = \cos x\,dx$. We then have

$$\int \frac{\cos x}{\sin^2 x + 1}\,dx \;=\; \int \frac{du}{u^2+1}\,du \;=\; \tan^{-1} u \;+\; C = \tan^{-1}(\sin x) \;+\; C\;.$$

(c) (10 points) $\displaystyle\int \frac{x^2}{\sqrt{x+1}}\,dx.$

Solution: Use the substitution $u = x+1$ to get $du = dx$. Since $x = u-1$, we have $x^2 = (u-1)^2 = u^2 - 2u + 1$. and so

$$\begin{aligned}
\int \frac{x^2}{\sqrt{x+1}}\,dx &= \int \frac{u^2-2u+1}{\sqrt{u}}\,du = \int u^{3/2} - 2u^{1/2} + u^{-1/2}\,du \\
&= \frac{u^{5/2}}{5/2} - 2\frac{u^{3/2}}{3/2} + \frac{u^{1/2}}{1/2} + C = \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + 2u^{1/2} + C \\
&= \frac{2}{5}(x+1)^{5/2} - \frac{4}{3}(x+1)^{3/2} + 2(x+1)^{1/2} + C
\end{aligned}$$

(2) Evaluate the following definite integrals.

(a) (20 points) $\displaystyle\int_0^1 (3 + x\sqrt{x})\,dx$

Solution:

$$\int_0^1 (3 + x\sqrt{x})\,dx \;=\; 3\int_0^1 dx \;+\; \int_0^1 x^{3/2}\,dx \;=\; 3 + \left[\frac{x^{5/2}}{5/2}\right]_0^1 + 0 \;=\; 3 + \frac{2}{5} \;=\; \frac{17}{5}\;.$$

(b) (20 points) $\displaystyle\int_1^{e^2} \frac{(\ln x)^2}{x}\,dx$

Solution: Use the substitution $u = \ln x$ to get $du = \frac{dx}{x}$. Since $u = 0$ when $x = 1$ and $u = 2$ when $x = e^2$, we have

$$\int_1^{e^2} \frac{(\ln x)^2}{x}\,dx = \int_0^2 u^2\,du \;=\; \left.\frac{u^3}{3}\right|_0^2 = \frac{8}{3}\;.$$

(c) (10 points) $\int_0^1 x^3\sqrt{1-x^2}\,dx$

Solution: Use the substitution $u = 1 - x^2$ to get $du = -2x\,dx$. With this substitution, $x^3\,dx = \frac{-1}{2}(x^2)(-2x\,dx) = \frac{-1}{2}(1-u)\,du$, and

$$u = 0 \quad \text{when} \quad x = 1 \quad \text{and} \quad u = 1 \quad \text{when} \quad x = 0.$$

Hence,

$$\begin{aligned}\int_0^1 x^3\sqrt{1-x^2}\,dx &= \frac{-1}{2}\int_1^0 (1-u)\sqrt{u}\,du = \frac{1}{2}\int_0^1 u^{1/2} - u^{3/2}\,du \\ &= \frac{1}{2}\left[\frac{u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right]_0^1 = \frac{1}{2}\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{2}{15}\,.\end{aligned}$$

(3) Differentiate the following functions.

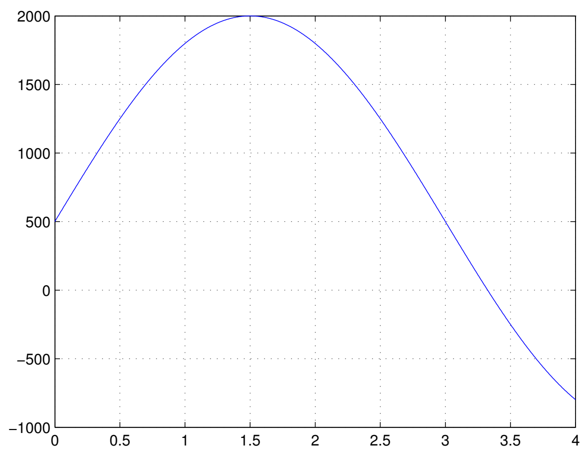
(a) (25 points) $F(x) = \int_1^{2x} \frac{t^2-1}{t^4+1}\,dt$

Solution: $F'(x) = 2\frac{(2x)^2-1}{(2x)^4+1}$.

(b) (25 points) $G(x) = \int_{\cos x}^{e^{x^2}} \sec u\,du$

Solution: $G'(x) = 2xe^{x^2}\sec(e^{x^2}) + \sin x\sec(\cos x)$.

(4) Water flows into and out of a storage tank. A graph of the rate of change $r(t)$ of the volume of the water in the tank, in liters per day, is shown. Assume the amount of water in the tank at time $t = 0$ is 5,000 L.



(a) (15 points) Use a Riemann Sum with $\Delta t = 1$ to give an upper bound for the amount of water in the take 4 days later.

Solution: Upper Bound = 20000 + 1800 + 2000 + 1800 + 500 = 26, 100 liters.

(b) (15 points) Use a Riemann Sum with $\Delta t = 1$ to give a lower bound for the amount of water in the take 4 days later.

Solution: Lower Bound $= 20000 + 500 + 1750 + 500 - 800 = 21950$.

(c) (20 points) Use the Midpoint Rule to estimate the amount of water in the take 4 days later.

Solution: Midpoint Rule Estimate $= 20000+1250+2000+1250-250 = 24250$.

(5) In this problems we consider the motion of a falling body with gravity being the only force acting on the body. Recall that the acceleration due to gravity is $-32\ ft/sec^2$.

(a) (20 points) If the initial velocity at time $t = 0$ is v_0 and the initial height above ground at time $t = 0$ is h_0, give a formula for $h(t)$ the height above ground at time t?

Solution:

Acceleration: $a(t) = -32\, ft/s^2$.

Velocity: $\int a(t)\, dt \;=\; -32t + C,\; v(0) = v_0 \;\Rightarrow\; C = v_0$.

Height: $h(t) = \int v(t)\, dt \;=\; \int -32t + v_0\, dt \;=\; -16t^2 + v_0 t + D$, $h(0) = h_0 \;\Rightarrow\; D = h_0$.

We have

$$v(t) = -32t + v_0 \qquad \text{and} \qquad h(t) = -16t^2 + v_0 t + h_0\ .$$

(b) (15 points) Assuming $v_0 \geq 0$, at what time does the falling body attain its maximum height above ground?

Solution: Since $h(t)$ is concave down ($h''(t) = -32$), $h(t)$ attains it maximum value when $0 = h'(t) = v(t) = -32t + v_0$. That is, the time t_m at which the maximum height is attained satisfies $0 = -32t_m + v_0$, or equivalently,

$$t_m = \frac{v_0}{32} \geq 0\ .$$

(c) (15 points) A ball is thrown up in the air from an initial height of h_0 feet. After 3 seconds it reaches its maximum height, and after 10 seconds it hits the ground. What is h_0?

Solution: By Part (b) above, we know that $3 = t_m = v_0/32$ so that $v_0 = 96$. By Part (a) we now know that $h(t) = -16t^2 + 96t + h_0$. Furthermore, at time $t = 10$ we have $h(10) = 0$, and so

$$0 = h(10) = -16(10)^2 + 96(10) + h_0 = -1600 + 960 + h_0 = -640 + h_0\ .$$

Therefore, $h_0 = 640\, ft$.