MATH 125G Second MIDTERM November 13, 2008
SOLUTIONS

(1) Evaluate the following indefinite integrals.

(a) $\displaystyle\int \mathsf{e}^x\sqrt{1+\mathsf{e}^x}\,dx$.

Solution: Use the substitution $u = 1+\mathsf{e}^x$ to get $du = \mathsf{e}^x\,dx$, and so

$$\int \mathsf{e}^x\sqrt{1+\mathsf{e}^x}\,dx \;=\; \int \sqrt{u}\,du \;=\; \frac{2}{3}u^{3/2}+C \;=\; \frac{2}{3}(1+\mathsf{e}^x)^{3/2}+C\ .$$

(b) $\displaystyle\int \frac{\ln x}{x^2}\,dx$.

Solution: Use integration by parts with

$$u = \ln x \qquad dv = x^{-2}\,dx$$
$$du = x^{-1}\,dx \qquad v = -x^{-1}\,dx$$

to get

$$\int \frac{\ln x}{x^2}\,dx \;=\; -\frac{\ln x}{x} + \int x^{-2}\,dx \;=\; -\frac{\ln x}{x} - x^{-1} + C\ .$$

(c) $\displaystyle\int \frac{x+1}{\sqrt{x^2+4x-5}}\,dx$.

Solution: First complete the square to get $x^2+4x-5 = (x+2)^2-9$, and so

$$\int \frac{x+1}{\sqrt{x^2+4x-5}}\,dx \;=\; \int \frac{x+1}{\sqrt{(x+2)^2-9}}\,dx\ .$$

Next use the trig. sub. $x+2 = 3\sec\theta$ with $dx = 3\sec\theta\tan\theta\,d\theta$ giving

$$\begin{aligned}
\int \frac{x+1}{\sqrt{(x+2)^2-9}}\,dx &= \int \frac{3\sec\theta - 1}{3\tan\theta}3\sec\theta\tan\theta\,d\theta \\
&= \int 3\sec^2\theta - \sec\theta\,d\theta \\
&= 3\tan\theta - \ln|\sec\theta + \tan\theta| + C \\
&= 3\sqrt{\left(\frac{x+2}{3}\right)^2 - 1} - \ln\left|\frac{x+2}{3} + \sqrt{\left(\frac{x+2}{3}\right)^2-1}\right| + C \\
&= \sqrt{x^2+4x-5} - \ln|x+2+\sqrt{x^2+4x-5}| + C\ .
\end{aligned}$$

(2) Evaluate the following definite integrals.

(a) $\displaystyle\int_{-1}^{1} \frac{1}{x^2+2x+5}\,dx$

Solution: First complete the square to get $x^2+2x+5=(x+1)^2+4$ giving

$$\int_{-1}^{1} \frac{1}{x^2+2x+5}\,dx = \int_{-1}^{1} \frac{1}{(x+1)^2+4}\,dx = \frac{1}{2}\tan^{-1}\frac{x+1}{2}\Bigg|_{-1}^{1} = \frac{\pi}{8}.$$

(b) $\displaystyle\int_{1}^{\sqrt{3}} \frac{x+1}{x^3+x}\,dx$

Solution: First compute the partial fractions decomposition

$$\frac{x+1}{x^3+x} = \frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1},$$

so that $x+1 = A(x^2+1)+(Bx+C)x$. Setting $x=0$ gives $A=1$, and so $-x+1 = Bx+C$ which implies that $B=-1$ and $C=1$. Therefore,

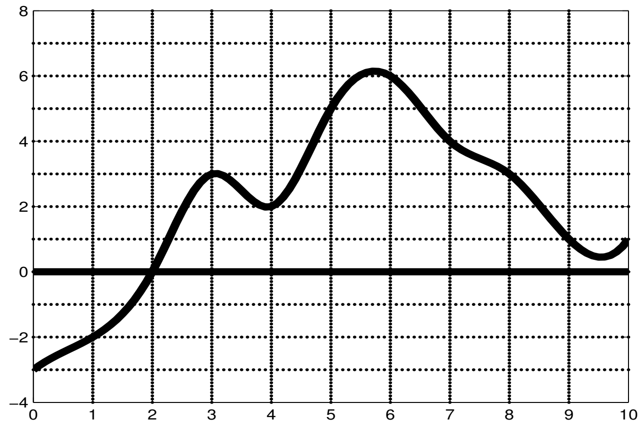
$$\begin{aligned}
\int_{1}^{\sqrt{3}} \frac{x+1}{x^3+x}\,dx &= \int_{1}^{\sqrt{3}} \frac{1}{x} + \frac{1}{x^2+1} - \frac{x}{x^2+1}\,dx \\
&= \ln x + \tan^{-1} x - \frac{1}{2}\ln(x^2+1)\Bigg|_{1}^{\sqrt{3}} \\
&= \ln\sqrt{3} + \frac{\pi}{3} - \ln 2 - \frac{\pi}{4} + \ln\sqrt{2} \\
&= \ln\sqrt{\frac{3}{2}} + \frac{\pi}{12}\ .
\end{aligned}$$

(c) $\displaystyle\int_{2}^{6} \frac{1}{x\sqrt{4x+1}}\,dx.$

Solution: Set $u=\sqrt{4x+1}$ so that $x=\frac{u^2-1}{4}$ and $dx=\frac{1}{2}u\,du$. Then (using a partial fractions decomposition for $\frac{2}{u^2-1}$)

$$\begin{aligned}
\int_{2}^{6} \frac{1}{x\sqrt{4x+1}}\,dx &= \int_{3}^{5} \frac{4}{(u^2-1)u}\,\frac{u}{2}\,du \\
&= \int_{3}^{5} \frac{2}{u^2-1}\,du \\
&= \int_{3}^{5} \frac{1}{u-1} - \frac{1}{u+1}\,du \\
&= \ln\left|\frac{u-1}{u+1}\right|\Bigg|_{3}^{5} \\
&= \ln\frac{4}{6} - \ln\frac{2}{4} = \ln\frac{4}{3}\ .
\end{aligned}$$

(3) Use Simpson's rule with $n = 8$ to estimate the volume of the solid obtained by rotating the function given below about the x-axis on the interval $2 \le x \le 10$.



Solution: Use the disk method to get

$$\begin{aligned}
\text{Volume} &= \int_2^{10} \pi r^2\, dx = \pi \int_2^{10} (f(x))^2\, dx \\
&= \frac{\pi}{3}\left[0^2 + 4\cdot 3^2 + 2\cdot 2^2 + 4\cdot 5^2 + 2\cdot 6^2 + 4\cdot 4^4 + 2\cdot 3^2 + 4\cdot 1^2 + 1^2\right] \\
&= 101\pi\ .
\end{aligned}$$

(4) (a) Compute the volume of the solid obtained by rotating the area between the two curves $y = x$ and $y = 1 - (x-1)^2$ about the y-axis.

Solution: The two curves intersect when $x = 1 - (x-1)^2$, or $x = 0, 1$. Using the shell method we get

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^1 rh\, dx = 2\pi \int_0^1 x(1 - (x-1)^2 - x)\, dx \\
&= 2\pi \int_0^1 x^2 - x^3\, dx \\
&= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}\ .
\end{aligned}$$

(b) Given the real number m, what are the points of intersection for the two curves $y = mx$ and $y = 1 - (x-1)^2$?

Solution: The two curves intersect when $mx = 1 - (x-1)^2 = -x^2 + 2x$, or $0 = x^2 - (2-m)x = x(x - (2-m))$, or $x =, 0, (2-m)$.

(c) For what value (or values) of m does the volume of the solid obtained by rotating the area between the two curves

$$y = mx \qquad \text{and} \qquad y = 1 - (x-1)^2$$

about the y-axis equal 16 times the volume obtained in part (a) (i.e. for $m = 1$).

Solution: Again using the shell method we get

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^{(2-m)} rh\,dx \;=\; 2\pi \int_0^{(2-m)} x(-x^2+2x-mx)\,dx \\
&= 2\pi \int_0^{(2-m)} (2-m)x^2 - x^3\,dx \\
&= 2\pi \left[(2-m)\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{(2-m)} \\
&= \frac{2\pi(2-m)^4}{12} = (2-m)^4\frac{\pi}{6}\ .
\end{aligned}$$

Hence the volume is 16 times that of part (a) when $2^4 = (2-m)^3$, or, equivalently, when $m = 0, 4$.

(5) Consider a circular flat bottomed pool of diameter 30 feet and depth 6 feet. The pool is currently filled with 5 feet of water. We have rented a 30 gallon per minute pump to empty the pool.

$$1\,\text{ft}^3 \approx 7.5 \text{ gallons} \quad \text{and} \quad 1\,\text{gallon} \approx 8\text{lbs}\ .$$

(a) (i) How many cubic feet of water can the pump remove from the pool in 3 hours, and (ii) how deep is the water in the pool after 3 hours of pumping?

Solution: (i) $30\,\frac{\text{gal}}{\text{min}} \cdot 180\,\text{min} \cdot \frac{1}{7.5}\,\frac{\text{ft}^3}{\text{gal}} \;=\; 720\,\text{ft}^3$.
(ii) Let $h =$ feet of water removed. Then $\pi 15^2 h = 720$ or $h = \frac{16}{5\pi}$ so water depth $= 5 - \frac{16}{5\pi}$.

(b) Assuming that the pump only needs to lift the water to the height of the edge of the pool, how much work is done in the first 3 hours of pumping the water out of the pool?

Solution: The weight of the water in Δx feet of the pool is

$$(\pi \cdot 15^2 \cdot \Delta x)\,\text{ft}^3 \cdot 7.5\,\frac{\text{gal}}{\text{ft}^3} \cdot 8\,\frac{\text{lbs}}{\text{gal}} \;=\; 2^2 3^3 5^3 \pi \Delta x\,\text{lbs}\ .$$

Hence

$$\begin{aligned}
\text{Work} &= \int_0^{16/(5\pi)} 2^2 3^3 5^3 \pi (1+x)\,dx \\
&= 2^2 3^3 5^3 \pi \left[x + \frac{x^2}{2}\right]_0^{2^4/(5\pi)} \\
&= 2^2 3^3 5^3 \pi \left[\frac{2^4}{5\pi} + \frac{1}{2}\frac{2^8}{5^2\pi^2}\right] \\
&= 2^6 3^3 5^2 \left[1 + \frac{8}{5\pi}\right] \\
&= 43200[1 + 8/(5\pi)]\ \text{lb-ft}\ .
\end{aligned}$$