An Inequality Constrained Nonlinear Kalman-Bucy Smoother

Bradley Bell [1] James Burke [2] Gianluigi Pillonetto [3]

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[1] Applied Physics Laboratory, University of Washington
[2] Mathematics, University of Washington
[3] Information Engineering, University of Padova

Applications

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- Aircraft tracking (location and velocities)

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- Aircraft tracking (location and velocities)
- Ship tracking

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- Underwater tracking

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- Aircraft tracking (location and velocities)
- Ship tracking
- Underwater tracking
- Compartmental analysis of drug concentrations and clearance

Outline

- Unconstrained Kalman-Bucy smoother.

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  An optimization viewpoint (Bell 94)

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- An interior point framework.

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- Unconstrained Kalman-Bucy smoother.

  An optimization viewpoint (Bell 94)

- Constrained Kalman-Bucy smoother.

- An interior point framework.

- A toy ship tracking problem.

- Implementation of a constrained iterated Kalman-Bucy smoother.

Unconstrained Transition Model

$$x_k = g_k(x_{k-1}) + w_k, \quad k = 1, 2, \ldots, N \text{ (time)}$$

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- $\hat{x}_0$ a known estimate, $e_0 \sim N(0, P_0)$

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- $w_k \sim N(0, Q_k)$ transition noise

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- $w_k \sim N(0, Q_k)$ transition noise
- $Q_k \in \mathcal{S}^n_+$ transition covariance (known)

Unconstrained Measurment Model

$$z_k = h_k(x_k) + v_k, \quad k = 1, 2, \ldots, N$$

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- $z_k \in \mathbb{R}^m$ measurement vector (known)
- $h_k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ the measurement function (known)

Unconstrained Measurment Model

$$z_k = h_k(x_k) + v_k, \quad k = 1, 2, \ldots, N$$

- $z_k \in \mathbb{R}^m$ measurement vector (known)
- $h_k : \mathbb{R}^n \to \mathbb{R}^m$ the measurement function (known)
- $v_k \sim N(0, R_k)$ measurement noise
- $R_k \in \mathcal{S}_+^n$ measurement covariance

Full Model

$$
\begin{aligned}
x_0 &= \hat{x}_0 + e_0 \\
x_k &= g_k(x_{k-1}) + w_k, \quad k = 1, 2, \ldots, N \\
z_k &= h_k(x_k) + v_k, \quad k = 1, 2, \ldots, N
\end{aligned}
$$

$$
e_0 \sim N(0, P_0), \quad w_k \sim N(0, Q_k), \quad v_k \sim N(0, R_k)
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\end{aligned}
$$

$$
e_0 \sim N(0, P_0), \quad w_k \sim N(0, Q_k), \quad v_k \sim N(0, R_k)
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Affine Case: $\nabla g = \text{constant}$ and $\nabla h = \text{constant}$

The Likelihood Function

$$\mathcal{L}(x_0, \ldots, x_N) \quad =$$

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$$\mathcal{L}(x_0, \ldots, x_N) \quad = \quad K \exp\left[-\frac{1}{2}(x_0 - \hat{x}_0)^T P_0^{-1}(x_0 - \hat{x}_0)\right]$$

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The Unconstrained Likelihood Problem

The negative log-likelihood function

$$\begin{aligned} L(x_0,\ldots,x_N) &= \frac{1}{2}(x_0-\hat{x}_0)^T P_0^{-1}(x_0-\hat{x}_0) \\ &+\frac{1}{2}\sum_{k=1}^{N} \begin{array}{c} (x_k-g_k(x_{k-1}))^T\, Q_k^{-1}\,(x_k-g_k(x_{k-1})) \\ + \\ (z_k-h_k(x_k))^T\, R_k^{-1}\,(z_k-h_k(x_k)) \end{array} \end{aligned}$$

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$$\text{minimize } L(x) \qquad \text{s.t.} \quad x \in \mathbb{R}^{nN}$$

The Iterated Kalman-Bucy Smoother (Bell 94)

Kalman-Filter

$$\min_{x_N} \min_{x_{N-1}} \ldots \min_{x_1} \min_{x_0} L(x_0, x_1, \ldots, x_N)$$

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$$L(x_0, \ldots, x_N) \quad = \quad \frac{1}{2}(x_0 - \hat{x}_0)^T P_0^{-1}(x_0 - \hat{x}_0)$$

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$$z_0 = x_0, \quad h_0(x) = x, \quad R_0 = I$$

The Iterated Kalman-Bucy Smoother (Bell 94)

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Kalman-Filter

$$\min_{x_N} \min_{x_{N-1}} \dots \min_{x_1} \min_{x_0} L(x_0, x_1, \dots, x_N)$$

$$\begin{aligned}
L(x_0, \dots, x_N) \quad = \quad & \frac{1}{2}(x_0 - \hat{x}_0)^T P_0^{-1}(x_0 - \hat{x}_0) \\
& + \frac{1}{2}\left(z_0 - h_0(x_0)\right)^T R_0^{-1}\left(z_0 - h_0(x_0)\right) \\
& + \frac{1}{2}\left(x_1 - g_1(x_0)\right)^T Q_1^{-1}\left(x_1 - g_1(x_0)\right) \\
& + \frac{1}{2}\sum_{k=1}^{N-1} \begin{array}{c} \left(z_k - h_k(x_k)\right)^T R_k^{-1}\left(z_k - h_k(x_k)\right) \\ + \\ \left(x_{k+1} - g_k(x_k)\right)^T Q_{k+1}^{-1}\left(x_{k+1} - g_k(x_k)\right) \end{array}
\end{aligned}$$

Basic Optimization Problem: Affine Case

$$\begin{aligned} \min_{(x,y,w)} \quad & (x-\hat{x})^T P^{-1}(x-\hat{x}) + (z-h(x))^T R^{-1}(z-h(x)) \\ & (y-g(x))^T Q^{-1}(y-g(x)) + f(y,w) \end{aligned}$$

Basic Optimization Problem: Affine Case

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Dimension Reduction: Affine Case

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where

$$U = [P^{-1} + \nabla h R^{-1} \nabla h^T]^{-1}, \qquad \widehat{P} = Q + \nabla g^T U \nabla g,$$

$$u = \hat{x} + U \nabla h R^{-1} (z - h(\hat{x})), \qquad \hat{y} = g(u).$$

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Recursive application gives the Kalman Filter yielding the optimal x_N.

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Recursive application gives the Kalman Filter yielding the optimal x_N.

The Kalman-Bucy smoother is the complete optimal solution obtained by back-solving first for x_{N-1}, then for $x_{N-2}, \ \ldots$.

Dimension Reduction: Affine Case

$$\begin{aligned}
L(x_0, \ldots, x_N) &= \frac{1}{2}(x_0 - \hat{x}_0)^T P_0^{-1}(x_0 - \hat{x}_0) + \frac{1}{2}\left(z_0 - h_0(x_0)\right)^T R_0^{-1}\left(z_0 - h_0(x_0)\right) \\
&+ \frac{1}{2}\left(x_1 - g_1(x_0)\right)^T Q_1^{-1}\left(x_1 - g_1(x_0)\right) \\
&+ \frac{1}{2}\sum_{k=1}^{N-1} \begin{array}{c} \left(z_k - h_k(x_k)\right)^T R_k^{-1}\left(z_k - h_k(x_k)\right) \\ + \\ \left(x_{k+1} - g_k(x_k)\right)^T Q_{k+1}^{-1}\left(x_{k+1} - g_k(x_k)\right) \end{array}
\end{aligned}$$

Dimension Reduction: Affine Case

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\begin{aligned}
L(x_0, \ldots, x_N) &= \frac{1}{2}(x_0 - \hat{x}_0)^T P_0^{-1}(x_0 - \hat{x}_0) + \frac{1}{2}\left(z_0 - h_0(x_0)\right)^T R_0^{-1}\left(z_0 - h_0(x_0)\right) \\
&\quad + \frac{1}{2}\left(x_1 - g_1(x_0)\right)^T Q_1^{-1}\left(x_1 - g_1(x_0)\right) \\
&\quad + \frac{1}{2}\sum_{k=1}^{N-1} \begin{array}{c} \left(z_k - h_k(x_k)\right)^T R_k^{-1}\left(z_k - h_k(x_k)\right) \\ + \\ \left(x_{k+1} - g_k(x_k)\right)^T Q_{k+1}^{-1}\left(x_{k+1} - g_k(x_k)\right) \end{array}
\end{aligned}
$$

$$
L_1(x_1, \ldots, x_N) = \frac{1}{2}(x_1 - \hat{x}_1)^T P_1^{-1}(x_1 - \hat{x}_1)
$$

Dimension Reduction: Affine Case

$$
\begin{aligned}
L(x_0, \ldots, x_N) &= \frac{1}{2}(x_0 - \hat{x}_0)^T P_0^{-1}(x_0 - \hat{x}_0) + \frac{1}{2}\left(z_0 - h_0(x_0)\right)^T R_0^{-1}\left(z_0 - h_0(x_0)\right) \\
&+ \frac{1}{2}\left(x_1 - g_1(x_0)\right)^T Q_1^{-1}\left(x_1 - g_1(x_0)\right) \\
&+ \frac{1}{2}\sum_{k=1}^{N-1} \begin{matrix} \left(z_k - h_k(x_k)\right)^T R_k^{-1}\left(z_k - h_k(x_k)\right) \\ + \\ \left(x_{k+1} - g_k(x_k)\right)^T Q_{k+1}^{-1}\left(x_{k+1} - g_k(x_k)\right) \end{matrix}
\end{aligned}
$$

$$
L_1(x_1, \ldots, x_N) = \frac{1}{2}(x_1 - \hat{x}_1)^T P_1^{-1}(x_1 - \hat{x}_1) + \frac{1}{2}\left(z_1 - h_1(x_1)\right)^T R_1^{-1}\left(z_1 - h_1(x_1)\right)
$$

Dimension Reduction: Affine Case

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\begin{aligned}
L(x_0,\ldots,x_N) \;&=\; \frac{1}{2}(x_0-\hat{x}_0)^T P_0^{-1}(x_0-\hat{x}_0) + \frac{1}{2}\left(z_0-h_0(x_0)\right)^T R_0^{-1}\left(z_0-h_0(x_0)\right) \\
&+\frac{1}{2}\left(x_1-g_1(x_0)\right)^T Q_1^{-1}\left(x_1-g_1(x_0)\right) \\
&+\frac{1}{2}\sum_{k=1}^{N-1} \begin{array}{c} \left(z_k-h_k(x_k)\right)^T R_k^{-1}\left(z_k-h_k(x_k)\right) \\ + \\ \left(x_{k+1}-g_k(x_k)\right)^T Q_{k+1}^{-1}\left(x_{k+1}-g_k(x_k)\right) \end{array}
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L_1(x_1,\ldots,x_N) \;&=\; \frac{1}{2}(x_1-\hat{x}_1)^T P_1^{-1}(x_1-\hat{x}_1) + \frac{1}{2}\left(z_1-h_1(x_1)\right)^T R_1^{-1}\left(z_1-h_1(x_1)\right) \\
&+\frac{1}{2}\left(x_2-g_2(x_1)\right)^T Q_2^{-1}\left(x_2-g_2(x_1)\right)
\end{aligned}
$$

Dimension Reduction: Affine Case

$$
\begin{aligned}
L(x_0,\ldots,x_N) \; &= \; \frac{1}{2}(x_0-\hat{x}_0)^T P_0^{-1}(x_0-\hat{x}_0) + \frac{1}{2}\left(z_0-h_0(x_0)\right)^T R_0^{-1}\left(z_0-h_0(x_0)\right) \\
&+\frac{1}{2}\left(x_1-g_1(x_0)\right)^T Q_1^{-1}\left(x_1-g_1(x_0)\right) \\
&+\frac{1}{2}\sum_{k=1}^{N-1} \begin{array}{c} \left(z_k-h_k(x_k)\right)^T R_k^{-1}\left(z_k-h_k(x_k)\right) \\ + \\ \left(x_{k+1}-g_k(x_k)\right)^T Q_{k+1}^{-1}\left(x_{k+1}-g_k(x_k)\right) \end{array}
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L_1(x_1,\ldots,x_N) \; &= \; \frac{1}{2}(x_1-\hat{x}_1)^T P_1^{-1}(x_1-\hat{x}_1) + \frac{1}{2}\left(z_1-h_1(x_1)\right)^T R_1^{-1}\left(z_1-h_1(x_1)\right) \\
&+\frac{1}{2}\left(x_2-g_2(x_1)\right)^T Q_2^{-1}\left(x_2-g_2(x_1)\right) \\
&+\frac{1}{2}\sum_{k=2}^{N-1} \begin{array}{c} \left(z_k-h_k(x_k)\right)^T R_k^{-1}\left(z_k-h_k(x_k)\right) \\ + \\ \left(x_{k+1}-g_k(x_k)\right)^T Q_{k+1}^{-1}\left(x_{k+1}-g_k(x_k)\right) \end{array}
\end{aligned}
$$

The Iterated Kalman-Bucy Smoother (Bell 94)

$$\text{minimize } \widehat{L}(x; y) \qquad \text{s.t.} \quad y \in \mathbb{R}^{nN}$$

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$$\begin{aligned} \widehat{L}_k(x; y) &= \frac{1}{2}(x_0 - \hat{x}_0)^T P_0^{-1}(x_0 - \hat{x}_0) \\ &+ \frac{1}{2}\sum_{k=1}^{N} \begin{array}{c} (y_k - \hat{g}_k(x_{k-1}; y_{k-1}))^T \, Q_k^{-1} \, (y_k - \hat{g}_k(x_{k-1}; y_{k-1})) \\ + \\ \left(z_k - \hat{h}_k(x_k; y_k)\right)^T R_k^{-1} \left(z_k - \hat{h}_k(x_k; y_k)\right) \end{array} \end{aligned}$$

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$$\begin{aligned}
\widehat{L}_k(x; y) &= \frac{1}{2}(x_0 - \hat{x}_0)^T P_0^{-1} (x_0 - \hat{x}_0) \\
&\quad + \frac{1}{2} \sum_{k=1}^{N} \begin{array}{c} (y_k - \hat{g}_k(x_{k-1}; y_{k-1}))^T \, Q_k^{-1} \, (y_k - \hat{g}_k(x_{k-1}; y_{k-1})) \\ + \\ \left(z_k - \hat{h}_k(x_k; y_k)\right)^T R_k^{-1} \left(z_k - \hat{h}_k(x_k; y_k)\right) \end{array}
\end{aligned}$$

where

The Iterated Kalman-Bucy Smoother (Bell 94)

$$\text{minimize } \widehat{L}(x; y) \qquad \text{s.t.} \quad y \in \mathbb{R}^{nN}$$

$$\widehat{L}_k(x; y) = \frac{1}{2}(x_0 - \hat{x}_0)^T P_0^{-1}(x_0 - \hat{x}_0) + \frac{1}{2}\sum_{k=1}^{N} \begin{array}{c} (y_k - \hat{g}_k(x_{k-1}; y_{k-1}))^T \, Q_k^{-1} \, (y_k - \hat{g}_k(x_{k-1}; y_{k-1})) \\ + \\ \left(z_k - \hat{h}_k(x_k; y_k)\right)^T R_k^{-1} \left(z_k - \hat{h}_k(x_k; y_k)\right) \end{array}$$

where

$$\hat{g}_k(x_{k-1}; y_{k-1}) = g_k(x_{k-1}) + g_k'(x_{k-1})(y_{k-1} - x_{k-1})$$

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&\quad + \frac{1}{2}\sum_{k=1}^{N} \begin{array}{c} (y_k - \hat{g}_k(x_{k-1}; y_{k-1}))^T \, Q_k^{-1} \, (y_k - \hat{g}_k(x_{k-1}; y_{k-1})) \\ + \\ \left(z_k - \hat{h}_k(x_k; y_k)\right)^T R_k^{-1} \left(z_k - \hat{h}_k(x_k; y_k)\right) \end{array}
\end{aligned}$$

where

$$\hat{g}_k(x_{k-1}; y_{k-1}) \;=\; g_k(x_{k-1}) + g_k'(x_{k-1})(y_{k-1} - x_{k-1})$$

$$\hat{h}_k(x_k; y_k) \;=\; h_k(x_k) + h_k'(x_k)(y_k - x_k)$$

A Large Scale Tridiagonal QP

$$\min \frac{1}{2} y^T C y + d^T y \qquad y \in \mathbb{R}^{nN}$$

A Large Scale Tridiagonal QP

$$\min \frac{1}{2} y^T C y + d^T y \qquad y \in \mathbb{R}^{nN}$$

$$C = \begin{bmatrix} C_1 & A_2^T & 0 & \\ A_2 & C_2 & A_3^T & 0 \\ 0 & \ddots & \ddots & \ddots \\ & 0 & A_N & C_N \end{bmatrix}$$

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Solution: $\bar{y} = -C^{-1} d$
Efficient tridiagonal solvers are available, but better algorithms are needed.

The Constrained Model

The transition model:

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\begin{aligned}
x_k &= g_k(x_{k-1}) + w_k \quad (k = 1, \ldots, N) \\
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$$f_k(x_{k-1}, x_k) \leq 0, \quad \text{where } f_k : \mathbb{R}^n \rightarrow \mathbb{R}^\ell \quad (k = 1, \ldots, N)$$

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Goal: Develop a solver that uses the same linear algebraic tools as the unconstrained problem.

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(Strongly PD tridiagonal systems)

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$$\begin{array}{ll} \text{minimize} & \frac{1}{2} y^T C y + d^T y \\ \text{subject to} & b + By + s = 0, \quad 0 \le s \end{array}$$

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$$B = \begin{bmatrix} D_1 f_1(x_1) & 0 & \cdots & 0 & 0 \\ D_1 f_2(x_1, x_2) & D_2 f_2(x_1, x_2) & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & D_{N-1} f_N(x_{N-1}, x_N) & D_N f_N'(x_{N-1}, x_N) \end{bmatrix}$$

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B is a lower triangular bi-diagonal matrix.

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We apply a predictor-corrector Newton method to follow this trajectory as $\mu \downarrow 0$.

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$t > 0$ is a line search parameter

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A Toy Ship tracking Example

Track a ship traveling close to shore.

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Track a ship traveling close to shore.

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- The location of the shoreline: The graph of $1.25 - \sin(t)$.

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  Thus, an off-shore constraint is given by

$$f(u, v) = 1.25 - \sin(u) - v \leq 0 \ .$$

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Dynamics

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$$X_2(t) = t \quad \text{first component of position}$$

$$X_4(t) = 1.3 - \sin(t) \quad \text{second component of position}$$

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$$X_1(t) = 1$$ first component of velocity

$$X_2(t) = t$$ first component of position

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Model the velocity error components as independent Brownian motions plus an initial velocity.

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Model the velocity error components as independent Brownian motions plus an initial velocity.

Model observed position as the integral of the velocity error plus an initial location.

Discrete Transition Model

Approximate the stochastic integral by Euler's method with constant spacing Δt. This gives the discrete process.

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with

$$Q_k = \begin{pmatrix} \Delta t & \Delta t^2/2 & 0 & 0 \\ \Delta t^2/2 & \Delta t^3/3 & 0 & 0 \\ 0 & 0 & \Delta t & \Delta t^2/2 \\ 0 & 0 & \Delta t^2/2 & \Delta t^3/3 \end{pmatrix} .$$

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Initialize with

$$g_1(x_0) = X(t_1)\ ,\ Q_1 = 100\,\mathrm{I}_{4\times 4}\ .$$

Measurement Model

Constant variance σ^2.

Shore distance measurement locations $(0, 0)$ and $(2\pi, 0)$.

$$h_k(x_k) = \begin{pmatrix} \sqrt{x_{2,k}^2 + x_{4,k}^2} \\ \\ \sqrt{(x_{2,k} - 2\pi)^2 + x_{4,k}^2} \end{pmatrix}$$

and

$$R_k = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} .$$

Constraints

The ship cannot cross land:

$$X_4(t) \geq 1.25 - \sin[X_2(t)]$$

so

$$f_k(x_k) = 1.25 - \sin(x_{2,k}) - x_{4,k} \ .$$

Initialization Data

$$N = 50 \qquad \text{number of measurement times}$$

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$\Delta t = 2\pi/N$		spacing between time points

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$$\begin{aligned}
N &= 50 && \text{number of measurement times} \\
\Delta t &= 2\pi/N && \text{spacing between time points} \\
t_k &= k\Delta t && \text{time corresponding to k-th measurement} \\
\sigma &= .25 && \text{standard deviation of measurement noise}
\end{aligned}$$

Initialization Data

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$\sigma = .25$		standard deviation of measurement noise
$v_k \sim N(0, R_k)$		simulated measurement noise

Initialization Data

N	$=$	50	number of measurement times
Δt	$=$	$2\pi/N$	spacing between time points
t_k	$=$	$k\Delta t$	time corresponding to k-th measurement
σ	$=$	$.25$	standard deviation of measurement noise
v_k	$\sim$	$N(0, R_k)$	simulated measurement noise
z_k	$=$	$h_k[X(t_k)] + v_k$	simulated measurement value

Initialization Data

$$
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\sigma &= .25 && \text{standard deviation of measurement noise}\\
v_k &\sim N(0, R_k) && \text{simulated measurement noise}\\
z_k &= h_k[X(t_k)] + v_k && \text{simulated measurement value}
\end{aligned}
$$

Initial state: $x_0 = (0, 0, 0, 1)^T$.

Initialization Data

$$
\begin{array}{lcll}
N & = & 50 & \text{number of measurement times} \\
\Delta t & = & 2\pi/N & \text{spacing between time points} \\
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v_k & \sim & N(0, R_k) & \text{simulated measurement noise} \\
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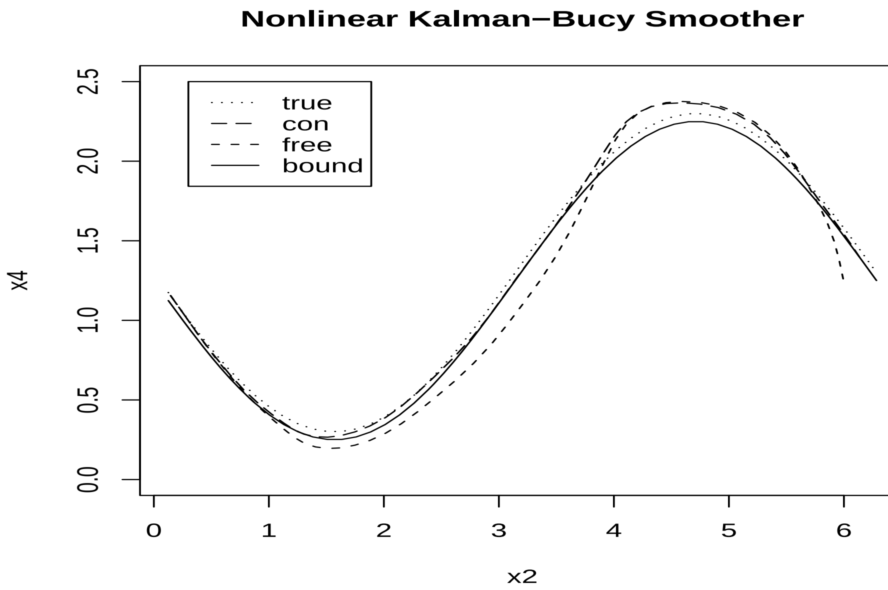
\end{array}
$$

Initial state: $x_0 = (0, 0, 0, 1)^T$.

Initial constraint violation:

$$f_k(x_k^0) = 1.25 - \sin(x_{2,k}) - x_{4,k} = .25 > 0$$



Nonlinear Kalman−Bucy Smoother
true
con
free
bound
x4
x2
0.0
0.5
1.0
1.5
2.0
2.5
0
1
2
3
4
5
6

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Bell, B.M., The marginal likelihood for parameters in a discrete Markov process, *IEEE Transactions on Signal Processing*, 48(2000) 870-873.

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The Nonlinear Algorithm

An SQP algorithm with ℓ_1 penalty linesearch.

$$\phi(x) = \sum_{k=1}^{N} \sum_{i=1}^{\ell} \max[f_k(x_k)_i, 0],$$

$$\tilde{\phi}(x; y) = \sum_{k=1}^{N} \sum_{i=1}^{\ell} \max[f_k(x_k)_i + f_k'(x_k)_i(y_k - x_k), 0]$$

Step 0: Initialization: Choose $x^0 \in \mathbb{R}^{nN}$ and $\alpha_0 > 0$. Set $\nu = 0$.

Step 1: Solve (inexactly) the QP subproblem for y^ν.

Step 2: Check convergence criteria.

Step 3: (Update the Penalty Parameter) Set $\hat{\alpha}_\nu = \alpha_\nu$. Define the value

$$\zeta_\nu = (y^\nu - x^\nu)^T C^\nu (y^\nu - x^\nu) + (a^\nu)^T (y^\nu - x^\nu)$$

If $\zeta_\nu \leq \hat{\alpha}_\nu \phi(x^\nu)$, set $\alpha_{\nu+1} = \hat{\alpha}_\nu$; otherwise, $\alpha_{\nu+1} = \max[\zeta_\nu / \phi(x^\nu)\ ,\ 2\hat{\alpha}_\nu]$.

Step 4: Compute the line search step size λ_p:

$$\begin{aligned}
\eta_\nu &= (a^\nu)^T(y^\nu - x^\nu) + \alpha_{\nu+1}[\tilde{\phi}(x^\nu; y^\nu) - \phi(x^\nu)] \\
H_\nu(\lambda) &= L[x^\nu + \lambda(y^\nu - x^\nu)] + \alpha_{\nu+1}\phi[x^\nu + \lambda(y^\nu - x^\nu)] \\
\lambda_n u &= \max\left\{2^{-q} \;\middle|\; q \in \mathbb{Z}_+ \text{ and } H_\nu(2^{-q}) - H_\nu(0) \le 2^{-q}\eta_\nu/10\right.
\end{aligned}$$

Step 5: Set $x^{\nu+1} = x^\nu + \lambda_\nu(y^\nu - x^\nu)$, then set $\nu = \nu + 1$ and go to Step 1.

Convergence Assumptions

1. The QP subproblems are all feasible. (Not strictly required)
2. The sequence $\{y^\nu\}$ is bounded.
3. The matrices $g_k'(x_{k-1}^\nu)$ are all invertible.
4. All cluster points of $\{x^\nu\}$ satisfy the generalized MFCQ.

Convergence Theorem

- The sequences
$$\{x^\nu\} \quad \text{and} \quad \{C_\nu\}$$
are bounded.
- $\{C_\nu\}$ is contained in a compact set of real symmetric positive definite matrices.
- Every cluster point is a KKT point.