When $T \in \mathcal{M}$, replacing φ by $\varphi \circ T$ does not change the integral in (6.2) and so we can assume

$$\operatorname{dist}(\varphi(0), \partial\Omega) \sim \operatorname{diam}(\Omega). \tag{6.9}$$

In that case we claim that if δ is sufficiently small, then every bad square is a bounded hyperbolic distance from $\varphi(0)$. By (6.9) and Lemma 6.3, that claim will prove the theorem.

Suppose S_j is a bad square such that $\varphi(z_0) \in S_j$ where $1 - |z_0|^2$ is small and where (6.3) holds at z_0. If δ is small, then by Lemma 4.3 there is a Möbius transformation

$$Tz = A\frac{z - z_1}{z - z_2}$$

such that

$$\big|T - \varphi\big| < \varepsilon \tag{6.10}$$

on the hyperbolic ball $B = \{z : \rho(z, z_0) \le b\}$ and

$$(1 - |z_0|^2)\left|\frac{T''(z_0)}{T'(z_0)}\right| \ge \frac{\varepsilon}{2}.$$

Hence by (4.10), the pole z_2 of T satisfies

$$\operatorname{dist}(z_2, \partial B) \le |z_2 - z_0| \le \frac{4}{\varepsilon}(1 - |z_0|^2),$$

while for $b = b(\varepsilon)$ fixed so that $\sinh(2b) = \varepsilon^{-2}$, ∂B has euclidian radius

$$r \ge \varepsilon^{-2}(1 - |z_0|) + \mathrm{O}\big((1 - |z_0|)^3\big).$$

Consequently, if $1 - |z_0|$ is small, there are adjacent arcs $I \subset \partial B$ and $J \subset \partial B$ with $\ell(I) = \ell(J) \ge \frac{2}{\sqrt{\varepsilon}}(1 - |z_0|)$ such that

$$\frac{\ell(T(I))}{\ell(T(J))} \ge \frac{C}{\varepsilon}.$$

But by (6.10), that contradicts (VII.2.5) for $\varphi(\partial B)$. ∎

7. The Bishop–Jones $H^{\frac{1}{2}-\eta}$ Theorem

When Ω is not a quasidisc the condition

$$\iint_{\mathbb{D}} |\varphi'(z)|\big|\mathcal{S}\varphi(z)\big|^2(1 - |z|^2)^3 dxdy < \infty \tag{7.1}$$

of Theorem 6.2 no longer implies that $\partial\Omega$ is rectifiable. For example, if Ω is a half-plane, then $\mathcal{S}\varphi = 0$ but $\partial\Omega$ is not rectifiable. However in [1994] Bishop and Jones obtained a sharp substitute theorem, and the proof of this theorem is the key to the deeper results in this chapter.

Theorem 7.1. *Let φ be the conformal mapping from $\mathbb{D}$ onto Ω. If*

$$B = |\varphi'(0)| + \iint\limits_{\mathbb{D}} |\varphi'(z)| \big|\mathcal{S}\varphi(z)\big|^2 (1-|z|^2)^3 dxdy < \infty,$$

then for any η, $0 < \eta < \frac{1}{2}$, $\varphi' \in H^{\frac{1}{2}-\eta}$, and

$$||\varphi'||_{H^{\frac{1}{2}-\eta}} \le C(\eta)B. \tag{7.2}$$

In particular, if the Bishop–Jones integral (7.1) is finite, then φ has non-zero angular derivative almost everywhere on $\partial\mathbb{D}$, the cone points of $\partial\Omega$ have full harmonic measure relative to Ω, and $\omega \ll \Lambda_1$ by Theorem VI.4.2. Theorem 7.1 is sharp; again the counterexample is the map φ from $\mathbb{D}$ to a half-plane.

For the applications to come, Theorem 7.1 will be less important than its local versions, Theorem 7.2 and Corollary 7.3. Recall we always write $g = \log(\varphi')$.

Theorem 7.2. *Let $E \subset \partial\mathbb{D}$ be compact. Let $U = \bigcup_{\zeta\in E}\Gamma_\beta(\zeta)$ be a cone domain, let $1 < \alpha < \beta$, and let $\varepsilon > 0$. Let φ be the conformal mapping from U onto a simply connected domain Ω and assume that at every $\zeta \in J \subset E$,*

$$\iint\limits_{\Gamma_\alpha(\zeta)} |\varphi'(z)| \big|\mathcal{S}\varphi(z)\big|^2 (1-|z|^2)^2 dxdy \le N < \infty. \tag{7.3}$$

Then there is $C(N,\varepsilon) < \infty$ and there exists $J_0 \subset J$ such that $|J_0| \ge (1-\varepsilon)|J|$ and

$$\iint\limits_{\Gamma_\alpha(\zeta)} |\varphi'(z)||g'(z)|^2 dxdy < C(N,\varepsilon) \tag{7.4}$$

at every $\zeta \in J_0$.

Corollary 7.3. *Let $E \subset \partial\mathbb{D}$ be compact. Let $U = \bigcup_{\zeta\in E}\Gamma_\beta(\zeta)$ be a cone domain, and let $1 < \alpha < \beta$. Let φ be the conformal mapping from U onto a simply connected domain Ω. Then at almost every $\zeta \in E$ for which*

$$\iint\limits_{\Gamma_\alpha(\zeta)} |\varphi'(z)| \big|\mathcal{S}\varphi(z)\big|^2 (1-|z|^2)^2 dxdy < \infty \tag{7.5}$$

we also have

$$\iint_{\Gamma_\alpha(\zeta)} |\varphi'(z)||g'(z)|^2 dxdy < \infty. \tag{7.6}$$

The corollary follows by sending $\varepsilon \to 0$ and $N \to \infty$ in Theorem 7.2. Since $F(z) = \big(\varphi'(z)\big)^{\frac{1}{2}}$ satisfies $|F'(z)|^2 = |\varphi'(z)||g'(z)|^2$, Theorem 1.3 and Corollary 7.3 imply φ has a non-zero angular derivative at almost every point where (7.5) holds.

The converse assertion, that (7.5) holds almost everywhere (7.6) holds, is very easy. Since $g = \log(\varphi')$ and $F = \big(\varphi'\big)^{\frac{1}{2}}$ we have $|\varphi'||g'|^2 = 4|F'|^2$, so that (7.6) implies

$$\iint_{\Gamma_\alpha(\zeta)} \big|F'(z)\big|^2 dxdy < \infty,$$

and by Lemma 1.5,

$$\iint_{\Gamma_\delta(\zeta)} \big|F''(z)\big|^2 (1-|z|)^2 dxdy < \infty$$

for any δ, $1 < \delta < \alpha$. Also recall from Section 6,

$$|\varphi'||\mathcal{S}\varphi|^2 \le 8|F''|^2 + 8|g'|^2|F'|^2.$$

Since $||g||_{\mathcal{B}} \le 6$, we conclude that

$$\iint_{\Gamma_\delta(\zeta)} |\varphi'(z)|\big|\mathcal{S}\varphi(z)\big|^2 (1-|z|^2)^2 dxdy < \infty$$

if (7.6) holds at ζ, and a point of density argument then implies (7.5) holds almost everywhere that (7.6) holds.

The proof of Theorem 7.2 resembles the main argument in the proof of Theorem 7.1 and we will prove Theorem 7.1 first and then indicate the changes needed to get Theorem 7.2.

Proof of Theorem 7.1.. We take $B = 1$. Set $\Gamma(\zeta) = \big\{z : |z-\zeta| < 1-|z|\big\}$. Given η, $0 < \eta < \frac{1}{2}$ and given $\lambda \ge \lambda_0 = \lambda_0(\eta) > 1$, we construct a region $\mathcal{R} \subset \mathbb{D}$ and a compact set $E \subset \partial\mathbb{D}$ such that

$$\bigcup_{\zeta\in E} \Gamma(\zeta) \subset \mathcal{R}, \tag{7.7}$$

$$\big|\partial\mathbb{D} \setminus E\big| \le C\lambda^{-1+2\eta}, \tag{7.8}$$

and

$$\iint_{\mathcal{R}} |\varphi'||g'|^2(1-|z|^2)dxdy \le C\lambda, \tag{7.9}$$

where the constant C in (7.8) and (7.9) depends on η but not on λ.

Assume that we have built sets $\mathcal{R}$ and E satisfying (7.7), (7.8), and (7.9). Write $p = \frac{1}{2} - \eta$ and as usual take $F = (\varphi')^{\frac{1}{2}}$. Then $\varphi' \in H^p$ if and only if $F \in H^{2p}$ and

$$||\varphi'||_{H^p} = ||F||^2_{H^{2p}}.$$

Recall the area function

$$\begin{aligned} AF(\zeta) &= \Big(\iint_{\Gamma(\zeta)} \big|F'(z)\big|^2 dxdy\Big)^{\frac{1}{2}} \\ &= \Big(\frac{1}{4}\iint_{\Gamma(\zeta)} |\varphi'(z)||g'(z)|^2 dxdy\Big)^{\frac{1}{2}}. \end{aligned}$$

By Theorem 1.1 $F \in H^{2p}$ if and only if $AF \in L^{2p}$ and

$$||F||_{H^{2p}} \le C_p ||AF||_{2p}.$$

Since $\big|\{\zeta : z \in \Gamma(\zeta)\}\big| \le c(1-|z|^2)$, we have by (7.7)

$$\int_E (AF)^2 d\theta \le C \iint_{\mathcal{R}} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy,$$

and thus by (7.8) and (7.9),

$$\begin{aligned} \big|\{\theta : AF(\theta) > t\}\big| &\le \big|\partial\mathbb{D} \setminus E\big| + \frac{1}{t^2}\int_E (AF)^2 d\theta \\ &\le C\lambda^{-1+2\eta} + C\frac{\lambda}{t^2}, \end{aligned} \tag{7.10}$$

for all $\lambda \ge \lambda_0$. Take $\lambda = t^{\frac{2}{2p+1}}$ so that the two terms on the extreme right of (7.10) are equal. Then by (7.10) there is $t_0 = t_0(\lambda_0)$ such that

$$\begin{aligned} ||AF||^{2p}_{2p} &= 2p\int_0^\infty t^{2p-1}\big|\{\theta : AF(\theta) > t\}\big|dt \\ &\le 2p\int_0^{t_0} 2\pi t^{2p-1}dt + C\int_{t_0}^\infty t^{\frac{4p^2-4p-1}{2p+1}}dt \\ &\le C(\eta) \end{aligned}$$

if $0 < \eta < 1$.

That proves (7.2), and the proof of Theorem 7.1 is reduced to constructing sets $\mathcal{R}$ and E that satisfy (7.7), (7.8), and (7.9).

We first construct $\mathcal{R}$. To start put $\{|z| < \frac{1}{2}\} \subset \mathcal{R}$ and notice that we have

$$\iint_{|z|<\frac{1}{2}} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy \leq C|\varphi'(0)| \tag{7.11}$$

because $||g||_{\mathcal{B}} \leq 6$.

As in Section 3 take the dyadic Carleson boxes

$$Q = \{re^{i\theta} : 1-2^{-n} \leq r < 1,\ \pi j 2^{-(n+1)} \leq \theta < \pi(j+1)2^{-(n+1)}\},$$

$0 \leq j < 2^{n+1}$, of sidelength $\ell(Q) = 2^{-n}$ and their top halves

$$\begin{aligned} T(Q) &= Q \cap \{1-2^{-n} \leq r < 1-2^{-(n+1)}\} \\ &= Q \setminus \bigcup\{Q' : Q' \subset Q, Q' \neq Q\}, \end{aligned}$$

and write z_Q for the center of $T(Q)$. Fix δ and ε to be determined later. Say $Q \in \mathcal{L}$, for **large**, if

$$\sup_{T(Q)} (1-|z|^2)^2 \big|\mathcal{S}\varphi(z)\big| > \delta. \tag{7.12}$$

When Q is large, define $\mathcal{D}(Q) = T(Q)$. Say $Q \in \mathcal{G}$, for **good**, if $Q \notin \mathcal{L}$ and

$$\sup_{T(Q)} (1-|z|^2)|g'(z)| < \varepsilon.$$

Say $Q \in \mathcal{B}$, for **bad**, if $Q \notin \mathcal{L}$ and Q is not good, i.e.,

$$\sup_{T(Q)} (1-|z|^2)|g'(z)| \geq \varepsilon.$$

If $Q \notin \mathcal{L}$, we call Q **maximal** if the next bigger $\widetilde{Q} \supset Q$, $\ell(\widetilde{Q}) = 2\ell(Q)$ satisfies $\widetilde{Q} \in \mathcal{L}$ or if $\ell(Q) = \frac{1}{2}$. Write $\mathcal{M}$ for the set of maximal $Q \notin \mathcal{L}$. When $Q \in \mathcal{M}$ define

$$\mathcal{D}(Q) = Q \setminus \bigcup\{Q' \subset Q,\ Q' \in \mathcal{L}\}.$$

Then

$$\Big\{z : \frac{1}{2} \leq |z| < 1\Big\} = \bigcup_{\mathcal{L}\cup\mathcal{M}} \mathcal{D}(Q), \tag{7.13}$$

and the sets under this union are disjoint.

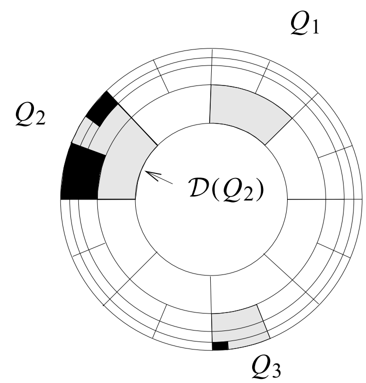

Figure X.20 $Q_1 \in \mathcal{L};\ Q_2, Q_3, \in \mathcal{M}$.

To complete the construction of $\mathcal{R}$ we consider four cases and we estimate the contribution to (7.9) in each case.

Case I: $Q \in \mathcal{L}$.

Put $\mathcal{D}(Q) = T(Q) \subset \mathcal{R}$ and pass to the next level of boxes $Q' \subset Q$ with $\ell(Q') = \frac{1}{2}\ell(Q)$. Since $||g||_{\mathcal{B}} \leq 6$, Theorem 4.1 and the Schwarz lemma yield

$$\begin{aligned}&\iint\limits_{T(Q)} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy\\&\leq C\Big(\frac{6}{\delta}\Big)^2 \iint\limits_{T(Q)} |\varphi'(z)|\big|\mathcal{S}\varphi(z)\big|^2(1-|z|^2)^3dxdy\end{aligned}\tag{7.14}$$

for every Case I box.

Case II: $Q \in \mathcal{G} \cap \mathcal{M}$.

Put $\mathcal{D}(Q) \subset \mathcal{R}$ and pass to the maximal $Q' \subset Q$. Recall that $F = \big(\varphi'\big)^{\frac{1}{2}}$ and $4|F'|^2 = |\varphi'||g'|^2$. To estimate the contribution to (7.9) in Case II we need the inequality

$$\begin{aligned}&\iint\limits_{\mathcal{D}(Q)} |\varphi'(z)|g'(z)|^2(1-|z|^2)dxdy\\&= 4\iint\limits_{\mathcal{D}(Q)} |F'(z)|^2(1-|z|^2)dxdy \leq C(\ell(Q))^3|F'(z_Q)|^2\\&\quad + C\iint\limits_{\mathcal{D}(Q)} |F''(z)|^2(1-|z|^2)^3dxdy.\end{aligned}\tag{7.15}$$

Because inequality (7.15) very much resembles (1.7) of Theorem 1.2 its proof is left as an exercise. Remembering that $2|F''| = |F||\mathcal{S}\varphi + (g')^2|$ and that

$||g||_{\mathcal{B}} \leq 6$, we obtain from (7.15)

$$\begin{aligned}\iint\limits_{\mathcal{D}(Q)} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy \\ \leq C\ell(Q)|\varphi'(z_Q)| + C\iint\limits_{\mathcal{D}(Q)} |\varphi'(z)|\big|\mathcal{S}\varphi(z)\big|^2(1-|z|^2)^3dxdy \\ + C\iint\limits_{\mathcal{D}(Q)} |\varphi'(z)||g'(z)|^4(1-|z|^2)^3dxdy\end{aligned} \tag{7.16}$$

in Case II. We need the following lemma.

Lemma 7.4. *Assume $\delta < \frac{\varepsilon^2}{2}$. If $Q \in \mathcal{G}$, if $Q' \subset Q$, and if $Q' \cap \mathcal{D}(Q) \neq \emptyset$, then $Q' \in \mathcal{G}$.*

Accept Lemma 7.4 for a moment. Then we can bound the last term in (7.16) using

$$\iint\limits_{\mathcal{D}(Q)} |\varphi'(z)||g'(z)|^4(1-|z|^2)^3dxdy \leq \varepsilon^2 \iint\limits_{\mathcal{D}(Q)} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy.$$

Therefore if $C\varepsilon^2 < 1$ we have

$$\begin{aligned}\iint\limits_{\mathcal{D}(Q)} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy \\ \leq C\ell(Q)|\varphi'(z_Q)| + C\iint\limits_{\mathcal{D}(Q)} |\varphi'(z)|\big|\mathcal{S}\varphi(z)\big|^2(1-|z|^2)^3dxdy.\end{aligned} \tag{7.17}$$

Consider the first term $C\ell(Q)|\varphi'(z_Q)|$ on the right side of (7.17). Because Q is maximal, either $\ell(Q) = 1/2$ or $Q \subset \widetilde{Q}$, $\ell(\widetilde{Q}) = 2\ell(Q)$ and (7.12) holds for $\widetilde{Q}$. In the first case

$$\ell(Q)|\varphi'(z_Q)| \leq c|\varphi'(0)|,$$

and this first case can occur for at most four squares Q. In the second case Theorem 4.1 and the Schwarz lemma give

$$\iint\limits_{T(Q)} \big|\mathcal{S}\varphi(z)\big|^2(1-|z|^2)^3dxdy \geq c\delta^2\ell(Q).$$

Because $||g||_{\mathcal{B}} \leq 6$ it follows that the first term on the right-hand side of (7.17) is majorized by a constant multiple of the second term, and hence that

$$\iint_{\mathcal{D}(Q)} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy \leq \frac{C}{\delta^2} \iint_{\mathcal{D}(Q)} |\varphi'(z)|\big|\mathcal{S}\varphi(z)\big|^2(1-|z|^2)^3 dxdy. \tag{7.18}$$

Thus for Case II boxes (7.18) holds with at most four exceptions, when we have the additional term $c|\varphi'(0)|$, and for (7.9) this additional term is harmless if $\lambda_0 \geq 1$.

Proof of Lemma 7.4. Let $r_1 e^{i\theta} \in T(Q')$. There is $r_0 \geq \frac{1}{2}$ with $r_0 e^{i\theta} \in T(Q)$ and $[r_0 e^{i\theta}, r_1 e^{i\theta}] \subset \mathcal{D}(Q)$. Set

$$r = \sup\big\{s \leq r_1 : (1-t^2)|g'(te^{i\theta})| \leq \varepsilon \text{ on } [r_0, s]\big\}.$$

Then since

$$(1-t^2)^2|S\varphi(te^{i\theta})| \leq \delta$$

(4.14) gives

$$|g''(te^{i\theta})| \leq \frac{\delta + \frac{\varepsilon^2}{2}}{(1-t^2)^2} < \frac{d}{dt}\Big(\frac{\varepsilon}{1-t^2}\Big)$$

on $[r_0, r]$. Therefore

$$(1-r^2)|g'(re^{i\theta})| < \varepsilon$$

and $r = r_1$. ■

The remaining two cases concern the bad boxes $Q \in \mathcal{B}$. We begin with a lemma that shows the bad boxes are sparsely distributed.

Lemma 7.5. *Given $\varepsilon > 0$ and an integer $n > 0$, there exist $C = C(\varepsilon)$ and $\delta = \delta(\varepsilon, n)$ such that if Q is a bad box for which*

$$\sup_{T(Q)} (1-|z|^2)^2|\mathcal{S}\varphi(z)| \leq \delta, \tag{7.19}$$

and if

$$B(Q) = Q \cap \{1-|z| \geq 2^{-n}\ell(Q)\} \cap \{(1-|z|^2)|g'(z)| \geq \varepsilon\},$$

then there exists a hyperbolic geodesic σ such that

$$\sup_{B(Q)} \rho(z,\sigma) < C,$$

where ρ denotes hyperbolic distance in $\mathbb{D}$. Moreover, given $\eta > 0$ there is $\delta > 0$ such that if (7.19) *holds for δ, then*

$$\Big(\frac{1-|z_0|}{1-|z_1|}\Big)^{2-\eta} \le \Big|\frac{\varphi'(z_1)}{\varphi'(z_0)}\Big| \le \Big(\frac{1-|z_0|}{1-|z_1|}\Big)^{2+\eta} \tag{7.20}$$

if $z_0 \in \sigma \cap T(Q)$ and $z_1 \in \sigma \cap Q \cap \{2^{n/2}\ell(Q) \le 1 - |z_1| \le 2^{-n}\ell(Q)\}$.

Proof. If δ is small, then by Lemma 4.3 there is a Möbius transformation T such that $|\varphi - T|$ is small on $Q \cap \{1 - |z| \ge 2^{-n}\ell(Q)\}$, so small in fact that

$$(1-|z|^2)^2\Big|\frac{T''(z)}{T'(z)} - g'(z)\Big| \le \frac{\varepsilon}{2}$$

on $Q \cap \{1 - |z| \ge 2^{-n}\ell(Q)\}$. Let $z_0 \in T(Q) \cap \{|g'(z)|(1 - |z|^2) \ge \varepsilon\}$, and let z^* be the pole of T. Then by (4.10) the conclusions of Lemma 7.5 all hold when σ is the circular arc or thogonal to $\partial\mathbb{D}$ that passes through z_0 and $\frac{z^*}{|z^*|}$, because they all hold for T.

■

Now fix $n \sim 10$ and $\beta = \beta(\varepsilon, \delta) \sim \delta^2$.

Case III: $Q \in \mathcal{B} \cap \mathcal{M}$ and

$$J = \iint\limits_{\mathcal{D}(Q)} |\varphi'(z)|\big|\mathcal{S}\varphi(z)\big|^2(1-|z|^2)^3 dxdy \ge \beta\ell(Q)|\varphi'(z_Q)|.$$

Let $\sigma = \sigma_Q$ be the geodesic of Lemma 7.5 and choose points $w_j \in \sigma \cap B(Q)$, $j = 0, 1, \dots, j_0$ such that $w_0 = z_0$ and $1 - |w_j| = 2^{-j}(1 - |z_0|)$. If possible, we let j^* be the least $j \le j_0$ such that

$$\sum_{j \le j^*}(1 - |w_j|)|\varphi'(w_j)| \sim \lambda J \ge \lambda\beta\ell(Q)|\varphi'(z_Q))|. \tag{7.21}$$

If j^* exists we take $Q^* = Q^*(Q) \subset Q$, $Q^* \in \mathcal{B}$, such that $w_{j^*} \in T(Q^*)$ and define $\widetilde{\mathcal{D}}(Q) = \mathcal{D}(Q) \setminus Q^*$. If j^* exists, then by (7.21) and the upper bound in (7.20),

$$\ell(Q^*(Q)) \le C\lambda^{-1+\eta}\ell(Q). \tag{7.22}$$

If no such j^* exists and if $w_{j_0} \in Q' \subset Q$ with $Q' \in \mathcal{B}$, we define $Q_1 = Q'$ and $\widetilde{\mathcal{D}}(Q) = \mathcal{D}(Q) \setminus Q_1$. If no such j^* exists and if no such Q' exists, we take $\widetilde{\mathcal{D}}(Q) = \mathcal{D}(Q)$. In each instance we put $\widetilde{\mathcal{D}}(Q) \subset \mathcal{R}$.

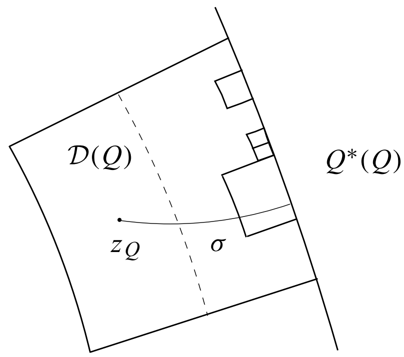

Figure X.21

By Lemmas 7.5 and 7.4 there is a cone Γ with vertex $\zeta \in \sigma \cap \partial\mathbb{D}$ such that if $Q' \in \mathcal{B}$, $Q' \subset Q$, and $Q' \cap \widetilde{\mathcal{D}}(Q) \neq \emptyset$, then there exists maximal dyadic $Q_j \in \mathcal{B}$, $Q_j \subset Q$, $Q_j \cap \widetilde{\mathcal{D}}(Q) \neq \emptyset$ such that $T(Q_j) \cap \Gamma \neq \emptyset$. Write

$$\iint\limits_{\widetilde{\mathcal{D}}(Q)} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy = \iint\limits_{\widetilde{\mathcal{D}}(Q)\setminus\Gamma} + \iint\limits_{\widetilde{\mathcal{D}}(Q)\cap\Gamma} .$$

Bt extending the radial edges of each maximal Q_j to Γ we partition $\widetilde{\mathcal{D}}(Q) \setminus \Gamma$ into chord-arc domains $\mathcal{D}_j \supset Q_j \cap \widetilde{\mathcal{D}}(Q) \setminus \Gamma$ with uniformly bounded chord-arc constants. Choose $z_j \in \Gamma \cap \partial\mathcal{D}_j$. Then by the proof for Case II and Harnack's inequality,

$$\begin{aligned}\iint\limits_{\widetilde{\mathcal{D}}(Q)} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy &= \sum \iint\limits_{\mathcal{D}_j} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy \\ &\quad + \iint\limits_{\widetilde{\mathcal{D}}(Q)\cap\Gamma} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy \\ &\leq \sum \iint\limits_{\mathcal{D}_j} |\varphi'(z)||\mathcal{S}\varphi(z)|^2(1-|z|^2)^3dxdy \\ &\quad + \sum (1-|z_j|^2)|\varphi'(z_j)| \\ &\quad + \iint\limits_{\widetilde{\mathcal{D}}(Q)\cap\Gamma} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy.\end{aligned}$$

The values $1-|z_j|$ decrease geometrically and each z_j is a bounded hyperbolic distance from the geodesic σ. Therefore by Harnack's inequality and (7.21),

$$\sum(1-|z_j|^2)|\varphi'(z_j)| \leq C\sum_{j\leq j^*}(1-|w_j|^2)|\varphi'(w_j)| \leq C\lambda J.$$

Harnack's inequality also gives

$$\iint\limits_{\widetilde{\mathcal{D}}(Q)\cap\Gamma} |\varphi'(z)||g'(z)|(1-|z|^2)dxdy \leq C\sum(1-|w_j|^2)|\varphi'(w_j)| \leq C\lambda J.$$

Since $\lambda > 1$, we therefore have

$$\begin{aligned}\iint\limits_{\widetilde{\mathcal{D}}(Q)} &|\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy \\ &\leq C\lambda \iint\limits_{\widetilde{\mathcal{D}}(Q)} |\varphi'(z)|\big|S\varphi(z)\big|^2(1-|z|^2)^3dxdy\end{aligned} \tag{7.23}$$

if Q is a Case III box and if j^* exists.

If no such j^* exists and if there is no $Q' \subset Q$ with $Q' \in \mathcal{B}$ and $w_{j_0} \in Q'$, we stop the sum at j_0 and we still obtain (7.23). Finally, if no j^* exists but if $w_{j_0} \in Q' \in \mathcal{B}$, $Q' \subset Q$, we repeat the construction with Q replaced by $Q_1 = Q'$ and with a possibly new geodesic σ_1 containing $w_{j_0} \in Q_1$, possibly constructing a new Q_1^* or a new $Q_1' = Q_2$, and we obtain (7.23) for $\widetilde{\mathcal{D}}(Q_1)$. We repeat the construction until we reach a case where a square Q_m^* is defined or a case where neither Q_m^* nor Q_m' is defined. If we reach a square Q_m^* we define $Q^*(Q) = Q_m^*$. Then (7.22) holds for $Q^*(Q)$. We put $\widetilde{\mathcal{D}}(Q) = \mathcal{D}(Q) \setminus Q^*(Q) = \bigcup \widetilde{\mathcal{D}}(Q_j)$ into $\mathcal{R}$. Note that (7.23) holds for each set $\widetilde{\mathcal{D}}(Q_j)$ and that these sets are disjoint.

Case IV: $Q \in \mathcal{B} \cap \mathcal{M}$ and

$$J = \iint\limits_{\mathcal{D}(Q)} |\varphi'(z)|\big|S\varphi(z)\big|^2(1-|z|^2)^3dxdy < \beta|\varphi'(z_Q)|\ell(Q).$$

Since $\beta \leq c\delta^2$, this case can only occur if $\ell(Q) = 1/2$, and thus for at most four boxes Q. Define Q^* by $Q^* \cap \sigma \neq \emptyset$ and

$$\ell(Q^*)|\varphi'(z_{Q^*})| \sim \lambda\ell(Q)|\varphi'(z_Q)| \tag{7.24}$$

and take $\widetilde{\mathcal{D}}(Q) = \mathcal{D}(Q) \setminus Q^*$. If no such Q^* exists take $\widetilde{\mathcal{D}}(Q) = \mathcal{D}(Q)$. Put $\widetilde{\mathcal{D}}(Q) \subset \mathcal{R}$ and do not consider any smaller $Q' \subset Q$. The argument used in

Case III yields

$$\iint_{\widetilde{\mathcal{D}}(Q)} |\varphi'(z)||g'(z)|^2(1-|z|^2)dxdy \leq C\lambda|\varphi'(0)|,$$

which is good enough because there are at most four such Q.

Proof of (7.7) and (7.9). By (7.13) we have

$$\mathcal{R} = \mathbb{D} \setminus \bigcup\{Q^*(Q) : Q \in \mathcal{B} \cap \mathcal{M}\}.$$

Since $|\varphi'(0)| \leq 6$, (7.11), (7.14), (7.18), (7.24), and the many cases of (7.23) give (7.9) for $\mathcal{R}$ provided $\lambda \geq B$. Let $I^*(Q) \subset \partial\mathbb{D}$ be the base of $Q^*(Q)$ and set

$$E = \partial\mathbb{D} \setminus \bigcup\{3I^*(Q) : Q \in \mathcal{B} \cap \mathcal{M}\}. \tag{7.25}$$

Then (7.7) holds on E.

Proof of (7.8). To prove (7.8) we need an additional lemma.

Lemma 7.6. *Given $\eta > 0$ there is $C = C(\eta)$ such that if $Q \in \mathcal{B} \cap \mathcal{M}$ and if $Q^*(Q)$ exists, then*

$$\ell(Q) \leq C\lambda^{\eta}\ell(\partial\mathcal{D}(Q) \cap \partial\mathbb{D}). \tag{7.26}$$

Proof of Lemma 7.6. By hypothesis and by (7.20) there exists Q^{**}, $Q^*(Q) \subset Q^{**} \subset Q$, such that

$$|\varphi'(z_{Q^{**}})|\ell(Q^{**}) \sim \lambda^{\eta}J, \tag{7.27}$$

and by the lower bound in (7.20)

$$\ell(Q^{**}) \geq C\lambda^{-\eta}\ell(Q). \tag{7.28}$$

Consider the chord-arc domain $\Omega = Q^{**} \cap \mathcal{D}(Q)$. By (7.27) and Corollary I.4.4, $\operatorname{diam}(\varphi(\Omega)) \geq C\lambda^{\eta}J$. By Theorem M.1 applied to $F = (\varphi')^{\frac{1}{2}}$ and by the proof of (7.22),

$$\ell(\varphi(\partial\Omega)) \leq C|\varphi'(z_{Q^{**}})|\ell(Q^{**}) + C\lambda J \leq C'|\varphi'(z_{Q^{**}})|\ell(Q^{**}).$$

Let $\mathcal{E} = \{Q' \in \mathcal{L} : \partial T(Q') \cap \partial\Omega \neq \emptyset\}$ and $A = \partial\Omega \cap \bigcup_{\mathcal{E}} \partial T(Q')$. Then by (7.12) and the Schwarz lemma,

$$\begin{aligned} \ell(\varphi(A)) &= \int_A |\varphi'(z)|ds \\ &\leq C\sum_{\mathcal{E}} \iint_{T(Q')} |\varphi'(z)|\big|S\varphi(z)\big|^2(1-|z|^2)^3dxdy \leq J. \end{aligned} \tag{7.29}$$

Since $\partial\varphi(\Omega)$ has length and diameter comparable to $\lambda^{\eta}J$ it follows from (7.29) and the Lavrentiev estimate (5.1) of Chapter VI that $\omega(z_{Q^{**}}, A, \Omega) = \omega(\varphi(z_{Q^{**}}, \varphi(A), \varphi(\Omega))$ is small provided $\lambda \geq \lambda_0(\eta)$. Then since Ω is a chord-arc domain with bounded constants, we conclude that $\frac{\ell(A)}{\ell(Q^{**})}$ is small and hence that

$$\ell(\partial\Omega \cap \partial\mathbb{D}) \geq c\ell(Q^{**}),$$

and with (7.28) this implies (7.26). ■

Finally, note that because the sets $\{\partial\mathcal{D}(Q) \cap \partial\mathbb{D} : Q \in \mathcal{B} \cap \mathcal{M}\}$ are pairwise disjoint, (7.26) and (7.22) give the inequality (7.8).

■

Proof of Theorem 7.2. A point of density argument shows that there exists $J_1 \subset J$, $|J_1| \geq (1 - \frac{\varepsilon}{3})|J|$ such that if $\mathcal{W} = \bigcup_{J_1} \Gamma_\beta(\zeta)$, then

$$\iint\limits_{\mathcal{W}} |\varphi'(z)||\mathcal{S}\varphi(z)|^2(1 - |z|^2)^3 dxdy$$
$$\leq C \int\limits_{J_1} \iint\limits_{\Gamma_\alpha(\zeta)} |\varphi'(z)||\mathcal{S}\varphi(z)|^2(1 - |z|^2)^2 dxdyds(\zeta) \leq CN.$$

Set

$$\mathcal{D} = \bigcup_{\zeta \in J_1} \Gamma_\beta(\zeta)$$

and

$$\mathcal{V} = \bigcup\{T(Q) : T(Q) \subset \mathcal{D}\}.$$

Then $\mathcal{V} \subset \mathcal{D}$. Define $\mathcal{L}$, $\mathcal{M}$, $\mathcal{G}$, and $\mathcal{B}$ as in the proof of Theorem 7.1, but include only those $T(Q)$ such that $T(Q) \subset \mathcal{D}$. For such Q define

$$\mathcal{D}(Q) = \mathcal{V} \cap Q \setminus \bigcup\{Q' \subset Q \cap \mathcal{D} : Q' \in \mathcal{L}\}.$$

Then the proof of Theorem 7.1 yields a set $E \subset J_1$, defined by (7.25) and a region $\mathcal{R} \subset \mathcal{V}$ so that (7.7), (7.8), and (7.9) hold, and so that $|E| \geq (1 - \frac{\varepsilon}{3})|J|$ and another point of density argument gives $J_0 \subset E$ for which $|J_0| \geq (1 - \varepsilon)|J|$ and for which (7.4) holds. ■

8. Schwarzians and BMO Domains

Recall that a simply connected domain Ω is called a BMO domain if the mapping function $\varphi : \mathbb{D} \to \Omega$ satisfies

$$g = \log(\varphi') \in \text{BMO}.$$

The results in this chapter yield two characterizations of BMO domains that complement Theorem VII.5.3.

Theorem 8.1. *The following are equivalent.*

(a) Ω *is a* BMO *domain.*

(b) *There exist* $\delta > 0$ *and* $C > 0$ *such that for all* $z_0 \in \Omega$ *there is a subdomain* $\mathcal{U} \subset \Omega$ *such that*

 (i) $z_0 \in \mathcal{U}$,

 (ii) $\partial\mathcal{U}$ *is rectifiable and* $\ell(\partial\mathcal{U}) \leq C\text{dist}(z_0, \partial\Omega)$, *and*

 (iii) $\omega(z_0, \partial\Omega \cap \partial\mathcal{U}, \mathcal{U}) \geq \delta$.

(c) *There exist* $\delta > 0$ *and* $C > 0$ *such that for all* $z_0 \in \Omega$ *there is a subdomain* $\mathcal{U} \subset \Omega$ *such that*

 (i) $z_0 \in \mathcal{U}$ *and* $\text{dist}(z_0, \partial\Omega) \leq C\,\text{dist}(z_0, \partial\mathcal{U})$,

 (ii) $\partial\mathcal{U}$ *is chord-arc with constant at most* C *and* $\ell(\partial\mathcal{U}) \leq C\,\text{dist}(z_0, \partial\Omega)$, *and*

 (iii) $\ell(\partial\Omega \cap \partial\mathcal{U}) \geq \delta\,\text{dist}(z_0, \partial\Omega)$.

(d) $|\mathcal{S}\varphi(z)|^2(1-|z|^2)^3dxdy$ *is a Carleson measure on* $\mathbb{D}$.

(e) *There exist* $\delta > 0$ *and* $C > 0$ *such that for every* $z_0 \in \mathbb{D}$ *there exists a Lipschitz domain* $\mathcal{V} \subset \mathbb{D}$ *such that*

 (i) $z_0 \in \mathcal{V}$,

 (ii) $\omega(z_0, \partial\mathcal{V} \cap \partial\mathbb{D}, \mathcal{V}) \geq \delta$, *and*

 (iii) $\displaystyle\iint_{\mathcal{V}} |\varphi'(z)||\mathcal{S}\varphi(z)|^2(1-|z|^2)^3dxdy \leq C|\varphi'(z_0)|(1-|z_0|^2)$.

Proof. Theorem VII.5.3 had the implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a). Here we use the arguments of the previous section to treat (a) $\Longleftrightarrow$ (d), (a) $\Longrightarrow$ (e), and (e) $\Longrightarrow$ (b).

(a) $\Longrightarrow$ (d): This was first observed by Zinsmeister [1984]. By (4.14) we have

$$|\mathcal{S}(\varphi)| \leq \frac{|g'|^2}{2} + |g''|.$$

It follows from (a) that $|g'(z)|^2(1-|z|^2)dxdy$ is a Carleson measure and hence $|g'(z)|^4(1-|z|^2)^3dxdy$ is also a Carleson measure, because $g \in \mathcal{B}$. For any analytic function we have

$$\iint_{T(Q)} |g''(z)|^2(1-|z|^2)^3 dxdy \le C \iint_{\widetilde{T}(Q)} |g'(z)|^2(1-|z|^2)dxdy,$$

where $\widetilde{T}(Q) = \{z : \text{dist}(z, T(Q)) \le \ell(Q)/4\}$. Thus $|g''(z)|^2(1-|z|^2)^3 dxdy$ is also a Carleson measure and (d) holds.

(d) $\Longrightarrow$ (a): This is due to Astala and Zinsmeister [1991]. Because (a) and (d) are invariant under conformal self maps of $\mathbb{D}$, to prove (a) it is enough to show

$$\int |g(e^{i\theta}) - g(0)|^2 d\theta \le C. \tag{8.1}$$

See Garnett [1981]. Let

$$A = \iint_{\mathbb{D}} |g'(z)|^2(1-|z|^2)dxdy$$

and

$$B = \iint_{\mathbb{D}} |g''(z)|^2(1-|z|^2)^3 dxdy.$$

Then since $|g'(0)| \le 6$ we have by Fourier series,

$$B \le 12A \le 3B + 6^3\pi \tag{8.2}$$

and by Theorem 1.2 it will be enough to show $B \le C'$.

By (4.14) we have

$$B \le 2\iint_{\mathbb{D}} |\mathcal{S}\varphi(z)|^2(1-|z|^2)^3 dxdy + \frac{1}{2}\iint_{\mathbb{D}} |g'(z)|^4(1-|z|^2)^3 dxdy. \tag{8.3}$$

Set $U = \{z \in \mathbb{D} : |g'(z)|(1-|z|^2) \le 1/2\}$. Then by (8.2)

$$\iint_U |g'(z)|^4(1-|z|^2)^3 dxdy \le \frac{A}{4} \le \frac{B}{12} + \frac{9}{2}\pi. \tag{8.4}$$

Set $V = \bigcup\{T(Q) : \sup_{T(Q)} |\mathcal{S}\varphi(z)|(1-|z|^2)^2 \ge \delta\}$ where $\delta = \delta(1/4, 10)$ from Lemma 7.5. Then because $||g||_{\mathcal{B}} \le 6$,

$$\iint_V |g'(z)|^4(1-|z|^2)^3 dxdy \le \frac{C}{\delta^2}\iint_V |\mathcal{S}\varphi(z)|^2(1-|z|^2)^3 dxdy. \tag{8.5}$$

If $T(Q) \setminus (U \cup V) \ne \emptyset$ then by Lemma 7.5 more than half of the $Q' \subset Q$ with $\ell(Q') = 2^{-10}\ell(Q)$ satisfy $T(Q') \subset (U \cup V)$. Consequently

$\mathbb{D} \setminus (U \cup V) \subset \bigcup T(Q_j)$, where $\{Q_j\}$ is a sequence of Carleson boxes with $\sum \ell(Q_j) \leq C''$, with C'' independent of φ. Hence

$$\iint\limits_{\mathbb{D}\setminus(U\cup V)} |g'(z)|^4(1-|z|^2)^3 dxdy \leq \sum_j C \iint\limits_{T(Q_j)} (1-|z|^2)^{-1} dxdy \leq C''. \tag{8.6}$$

Together (8.3), (8.4), (8.5), and (8.6) give us

$$\frac{23}{24} B \leq \left(2 + \frac{C}{\delta^2}\right) \iint\limits_{\mathbb{D}} |\mathcal{S}\varphi(z)|^2(1-|z|^2)^3 dxdy + \frac{9}{4}\pi + C'',$$

which establishes (8.1).

(d) and (a) $\Longrightarrow$ (e): Because (e) is invariant under Möbius self maps of $\mathbb{D}$, we can suppose $z_0 = 0$. Then let $\mathcal{V}$ be the Lipschitz region constructed in the proof of (a) $\Longrightarrow$ (b) from Theorem VII.5.3 and note that $|\varphi'|$ is bounded above and below on $\mathcal{V}$. Then use (d).

(e) $\Longrightarrow$ (b): We may assume $z_0 = \varphi(0)$. We repeat the proof of Theorem 7.1, with $\mathbb{D}$ replaced by the Lipschitz domain $\mathcal{V}$ given by (e), just as we did in the proof of Theorem 7.2. We obtain a Lipschitz domain $\mathcal{R} \subset \mathcal{V}$ such that $\int_{\partial\mathcal{R}} |\varphi'| ds < \infty$, with $0 \in \mathcal{R}$ and with $\omega(0, \partial\mathbb{D} \cap \partial\mathcal{R}, \mathcal{R}) \geq \frac{\delta}{2}$. Then (b) holds for $\mathcal{U} = \varphi(\mathcal{R})$. ■

9. Angular Derivatives

Let φ be a conformal mapping from $\mathbb{D}$ onto a simply connected domain Ω and let

$$G = \{\zeta \in \partial\mathbb{D} : \varphi \text{ has an angular derivative at } \zeta \text{ and } |\varphi'(\zeta)| \neq 0\}.$$

In this section we give several almost everywhere characterizations of G. By Theorem VI.6.1 we already know that $\zeta \in G$ is almost everywhere equivalent to

$$\varphi(\zeta) \text{ is a cone point of } \Omega. \tag{9.1}$$

Furthermore, for any $\alpha > 1$, $\zeta \in G$ is almost everywhere equivalent to

$$\iint\limits_{\Gamma_\alpha(\zeta)} \left|\varphi''(z)\right|^2 dxdy < \infty, \tag{9.2}$$

by the theorem of Marcinkiewicz, Zygmund, and Spencer, Theorem 1.3 above. Bishop and Jones [1994] gives several other almost everywhere characteriza-