Artin-Schelter regular algebras and the Steenrod algebra

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Exercise

Let $A(1)$ be the sub-Hopf algebra of the mod 2 Steenrod algebra generated by Sq^1 and Sq^2. Fill in this table:

Algebra	# of gens
$k[x,y]/(x^2,y^2)$	2
$A(1)$	?

Okay, that's easy, but perhaps it should be this table:

Algebra	?
$k[x,y]/(x^2,y^2)$	2
$A(1)$	?

Rephrasing: What is the generalization, when passing from finite-dimensional commutative algebras to finite-dimensional non-commutative algebras, of "the number of generators"?

Definition of AS regular algebra

Artin-Schelter regular algebras: Let k be a field. A k-algebra A is called Artin-Schelter regular if it is graded connected and the following three conditions hold:

- A has finite global dimension d,
- A is *Gorenstein*:

$$\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$$

- A has finite polynomial growth: there is positive number c such that $\dim A_n < c\, n^c$ for all $n \geq 0$.

Examples of AS regular algebras

Every AS regular algebra has a dimension, the integer d in the definition. AS regular algebras of dimensions ≤ 3 have been classified, and those of dimension 4 have been investigated.

Examples

- If $A = k[x_1, \dots, x_d]$, then A is AS regular of dimension d.
- If g is a finite-dimensional Lie algebra of dimension d, then its enveloping algebra $U\mathfrak{g}$ is AS regular of dimension d.

AS regular algebras: view as non-commutative analogue of polynomial algebras.

(Noetherian) + (AS regular) + (dim ≤ 4) $\Longrightarrow$ integral domain

Homomorphic images?

Question

What are the homomorphic images of AS regular algebras?

- The class includes all finite-dimensional algebras. (Use enveloping algebras.)
- If A is commutative, then A is the image of an AS regular algebra if and only if A is finitely generated.
- Thus, the question is really: What is the non-commutative generalization of "finitely generated commutative algebra"?
- No known alternate characterization.

We'll focus on the finite-dimensional case.

Any finite-dimensional algebra B has an AS regular algebra R with

- $R \twoheadrightarrow B$,
- if $\dim B = d$, then R has dimension d.

Can we do better – that is, find an AS regular algebra of smaller dimension – by focusing on certain families of finite-dimensional algebras?

A theorem

Theorem

Let B be a finite-dimensional sub-Hopf algebra of the mod 2 Steenrod algebra, with $\dim B = d$. Then there is an AS regular algebra of dimension $\log_2 d$ mapping onto it.

Odd primes: have partial analogue, maybe more.

The dimension $\log_2 d$ ought to be minimal:

Conjecture

Fix B as in the theorem with $\dim B = d$. If R is AS regular with $R \twoheadrightarrow B$, then R has dimension at least $\log_2 d$.

Question: If B is any finite-dimensional Hopf algebra over a field of characteristic p, with $\dim B = d$, is there an AS regular algebra of dimension $\log_p d$ mapping onto it?

Yes if the Hopf algebra satisfies a certain technical condition, unknown otherwise. More on this later.

Examples

At the prime 2:

Algebra	$d = \dim$	$\log_2 d$
$A(0)$	2	1
$A(1)$	8	3
$A(2)$	64	6
$A(3)$	1024	10
$A(n)$	$2^{(n+1)(n+2)/2}$	$(n+1)(n+2)/2$

Questions

- If B is a finite-dimensional Hopf algebra, is there an AS regular Hopf algebra mapping onto B by a Hopf algebra map? (We've been unable to find one for $A(2)$.)
- If B is the homomorphic image of an AS regular algebra, then it has a numerical invariant: the minimal dimension of an AS regular algebra mapping onto it. Applications? Other interpretations?
- In the commutative case, this invariant just gives the minimal number of generators.
- So we get the question from the start: What is the generalization, when passing from finite-dimensional commutative algebras to finite-dimensional non-commutative algebras, of "the number of generators"?

Theorem

Let B be a finite-dimensional sub-Hopf algebra of the mod 2 Steenrod algebra, with $\dim B = d$. Then there is an AS regular algebra of dimension $\log_2 d$ mapping onto it.

Proof.

- Filter B by powers of the augmentation ideal.
- The associated graded $\operatorname{gr} B$ is isomorphic to the restricted enveloping algebra of a restricted Lie algebra $\mathfrak{g}$ with trivial restriction.
- Use basis, generators, and relations for $\operatorname{gr} B \cong u\mathfrak{g}$ to write basis, generators, and relations for B.
- Throw out the restriction relations: Use generators and the "commutator relations" for B to define a new algebra A so that $\operatorname{gr} A \cong U\mathfrak{g}$.
- If $\operatorname{gr} A$ is AS regular, so is A, with the same dimension.

□

Example

$A(2)$, the algebra generated by Sq^1, Sq^2, and Sq^4.
Then $\mathrm{gr}\, A(2) \cong u\mathfrak{g}$ where $\mathfrak{g}$ is the 6-dimensional restricted Lie algebra defined as follows:

- The restriction is trivial.
- Basis: s_1, s_2, s_4 (the elements Sq^1, Sq^2, Sq^4),
- $s_{12} = [s_1, s_2]$, $s_{24} = [s_2, s_4]$, $s_{124} = [s_{12}, s_4]$,
- all other brackets trivial, except $s_{124} = [s_1, s_{24}]$.

We can write $A(2)$ as follows:

$$A(2) \cong k\langle s_1, s_2, s_4\rangle/(\text{relations}),$$

with s_{12}, s_{24}, and s_{124} defined as above, and relations of two types:

Example (continued)

$A(2) \cong k\langle s_1, s_2, s_4 \rangle / (\text{relations})$.

Commutator relations:

$$[s_1, s_4] = s_2 s_{12},\ [s_1, s_{24}] = s_{124},\ [s_2, s_{24}] = s_1 s_{124},\ [s_4, s_{24}] = s_2 s_1 s_{124},$$

and all other brackets trivial.

Restriction relations:

$$s_1^2 = 0,\ s_2^2 = s_1 s_{12},\ s_4^2 = s_2 s_{24},\ s_{12}^2 = 0 = s_{24}^2 = s_{124}^2.$$

The AS regular algebra $R(2)$ mapping onto $A(2)$ is obtained by deleting the restriction relations.

Applications

- An application of the proof: a family of bases, called commutator bases or PBW bases. The basis elements are ordered monomials in the iterated commutators (i.e., elements like s_1, s_{12}, s_{124}, ...). For example, a basis for $A(1)$:

$$\{1,\ s_1,\ s_2,\ s_{12},\ s_1 s_2,\ s_1 s_{12},\ s_2 s_{12},\ s_1 s_2 s_{12}\}.$$

- Understanding Ext? For example, we have

$$\begin{aligned}
A(1) &\cong k\langle s_1, s_2\rangle/([s_1, s_{12}], [s_2, s_{12}], s_1^2, s_2^2 + s_1 s_{12}, s_{12}^2), \\
R(1) &\cong k\langle s_1, s_2\rangle/([s_1, s_{12}], [s_2, s_{12}]), \\
R(1) &\twoheadrightarrow R(1)/(s_{12}^2) \\
&\twoheadrightarrow R(1)/(s_{12}^2, s_1^2) \\
&\twoheadrightarrow R(1)/(s_{12}^2, s_1^2, s_2^2 + s_1 s_{12}).
\end{aligned}$$