Cohomology operations and the Steenrod algebra

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Cohomology operations

cohomology operations $= \mathrm{NatTransf}(H^n(-;G), H^m(-;G'))$.
If X is a CW complex, then

$$H^n(X;G) \cong [X, K(G,n)].$$

So by Yoneda's lemma, there is a bijection

$$\begin{aligned} &\mathrm{NatTransf}(H^n(-;G), H^m(-;G')) \\ &\qquad \longleftrightarrow [K(G,n), K(G',m)] \\ &\qquad \cong H^m(K(G,n);G'). \end{aligned}$$

Thus elements of $H^m(K(G,n);G')$ give cohomology operations.

Serre, Borel, Cartan, et al. computed the groups $\widetilde{H}^m(K(G, n); G')$ for G, G' finite abelian.

First, note that they're zero when $m < n$ (by the Hurewicz theorem).

Now focus on case $G = \mathbf{Z}/p\mathbf{Z} = G'$, with p a prime.

The groups stabilize: for all q, n, there is a map

$$H^{q+n}(K(\mathbf{Z}/p\mathbf{Z}, n); \mathbf{Z}/p\mathbf{Z}) \to H^{q+n-1}(K(\mathbf{Z}/p\mathbf{Z}, n-1); \mathbf{Z}/p\mathbf{Z}),$$

It's an isomorphism when $q < n - 1$.

Iterate this. The inverse limit is the collection of mod p stable cohomology operations of degree q. Assemble together for all q: you get the mod p Steenrod algebra, which is an $\mathbf{F}_p$-algebra under composition.

The mod 2 Steenrod algebra A

- For any space (or spectrum) X, $H^*(X; \mathbf{F}_2)$ is a module over A.
- A is generated as an algebra by elements Sq^q (pronounced "square q"), with $\mathrm{Sq}^q : H^n(-) \to H^{n+q}(-)$.
- If X is a space:
  - $\mathrm{Sq}^q : H^q X \to H^{2q} X$ is the cup-squaring map.
  - $\mathrm{Sq}^q : H^i X \to H^{i+q} X$ is zero if $i < q$.
- A is associative, non-commutative. (Example: $\mathrm{Sq}^1 \mathrm{Sq}^2 \neq \mathrm{Sq}^2 \mathrm{Sq}^1$. On the polynomial generator x of $H^*(\mathbf{R}P^\infty)$, $\mathrm{Sq}^2 \mathrm{Sq}^1(x) = x^4$ while $Sq^1 Sq^2(x) = 0$.)

Applications

- Two spaces can have the same cohomology rings but different module structures over the Steenrod algebra, in which case they can't be homotopy equivalent. (Example: $\Sigma\mathbf{C}P^2$ and $S^3 \vee S^5$.)
- The Hopf invariant one problem: a nice multiplication on $\mathbf{R}^n$ $\rightsquigarrow$ a CW complex with mod 2 cohomology

  1 $\bullet$ x $\bullet$ (dim n) x^2 $\bullet$

  Hence $\mathrm{Sq}^n(x) \neq 0$ while $\mathrm{Sq}^i(x) = 0$ for $0 < i < n$. Thus Sq^n must be indecomposable in the mod 2 Steenrod algebra. This implies that n is a power of 2. (Adams refined this approach to solve the problem completely: $n = 1, 2, 4, 8$.)
- See Mosher-Tangora for more details and examples.

The Adams spectral sequence

Fix a prime p and let A be the mod p Steenrod algebra.
For spaces or spectra X and Y, there is a spectral sequence, the Adams spectral sequence, with

$$E_2 \cong \mathrm{Ext}_A^*(H^*Y, H^*X) \Longrightarrow [X, Y].$$

It “converges” if X and Y are nice enough.

Other topics:

- A is a graded Hopf algebra.
- Milnor's theorem: the graded vector space dual A_* of A has a very nice structure. At the prime 2: as algebras, $A_* \cong \mathbf{F}_2[\xi_1, \xi_2, \xi_3, \dots]$, and there is a simple formula for the comultiplication on each ξ_n.
- You can do computations in A using Sage.
- For generalized homology theories, it is often better to work with homology rather than cohomology: if E is a spectrum representing a homology theory, then E_*E is often better behaved than E^*E.
- For spectra X and Y, the Adams spectral sequence looks like

$$E_2 \cong \operatorname{Ext}^*_{E_*E}(E_*X, E_*Y),$$

abutting to $[X, Y]$.

Fix a prime p and let A be the mod p Steenrod algebra. Mod p cohomology defines a functor

$$\text{Spectra}^{\text{op}} \to A\text{-Mod}.$$

Make a new category, $A\text{-}\widetilde{\text{Mod}}$: same objects as A-Mod, but the morphisms from M to N are $\text{Ext}_A^*(M, N)$. Then we have functors

$$\text{Spectra}^{\text{op}} \to A\text{-Mod} \to A\text{-}\widetilde{\text{Mod}}$$

as well as a connection, via the Adams SS,

$$A\text{-}\widetilde{\text{Mod}} \rightsquigarrow \text{Spectra}^{\text{op}}.$$

So via cohomology and the Adams SS, the category $A\text{-}\widetilde{\text{Mod}}$ is an approximation to the category of spectra.

Furthermore, $A\text{-}\widetilde{\mathrm{Mod}}$ (actually a "fattened up" version of this category) has many formal similarities to Spectra: it satisfies the axioms for a stable homotopy category.
Some details:

- $A_* =$ graded dual of the mod p Steenrod algebra.
- $\mathrm{Ch}(A_*) =$ category with objects cochain complexes of A_*-comodules, morphisms cochain maps. Then $\mathrm{Ch}(A_*)$ has a (cofibrantly generated) model category structure.
- Cofibrations: degree-wise monomorphisms. Fibrations: degree-wise epimorphisms with degree-wise injective kernel. Weak equivalences: maps $f : X \to Y$ which induce an isomorphism

$$[\Sigma^i \mathbf{F}_p, J \otimes X] \to [\Sigma^i \mathbf{F}_p, J \otimes Y],$$

where J is an injective resolution of the trivial module $\mathbf{F}_p$.

The associated homotopy category is a stable homotopy category. Call it Stable(A_*).

Alternative construction

Stable(A_*) is the category with objects cochain complexes of injective A_*-comodules, morphisms cochain homotopy classes of maps.

- Smash product: $- \otimes_{\mathbf{F}_p} -$
- Sphere object: injective resolution of $\mathbf{F}_p$

There is a functor, in fact the inclusion of a full subcategory,

$$A_*\text{-Comod} \to A\text{-Mod}.$$

If M and N are A_*-comodules, then

$$\mathrm{Hom}_{\mathrm{Stable}(A_*)}(\Sigma^i M, N) \cong \mathrm{Ext}^i_A(M, N).$$

So cohomology gives us a functor

$$\mathrm{Spectra} \to \mathrm{Stable}(A_*)$$

and the Adams spectral gives a loose connection

$$\mathrm{Stable}(A_*) \rightsquigarrow \mathrm{Spectra}.$$

Furthermore, $\mathrm{Stable}(A_*)$ is a stable homotopy category.

Theorem (Nishida's theorem)

If $n > 0$, then every $\alpha \in \pi_n(S^0)$ is nilpotent.

Analogue for the Steenrod algebra – things are more complicated:

Theorem

Let $p = 2$. There is a ring R and a ring map $\mathrm{Ext}^*_A(\mathbf{F}_2, \mathbf{F}_2) \to R$ which is an isomorphism mod nilpotence.

The ring R can be described explicitly.
(This is also analogous to the Quillen stratification theorem for group cohomology.)

Idea of proof.

- Stable(A_*) is a stable homotopy category and $\mathbf{F}_2$ is the sphere object. So $\mathrm{Ext}_A^*(\mathbf{F}_2, \mathbf{F}_2)$ is the "homotopy groups of spheres".
- So there are Adams SS converging to $\mathrm{Ext}_A^*(\mathbf{F}_2, \mathbf{F}_2)$.
- That is:

$$? \Longrightarrow \mathrm{Ext}_A^*(\mathbf{F}_2, \mathbf{F}_2) \Longrightarrow \pi_*(S^0)$$

- In particular: any Lyndon-Hochschild-Serre spectral sequence can be viewed as an Adams spectral sequence.
- For a certain normal sub-Hopf algebra $D \leq A$:

$$\mathrm{Ext}_{A//D}^*(\mathbf{F}_2, \mathrm{Ext}_D^*(\mathbf{F}_2, \mathbf{F}_2)) \Longrightarrow \mathrm{Ext}_A^*(\mathbf{F}_2, \mathbf{F}_2).$$

  $\mathrm{Ext}_A^*(\mathbf{F}_2, \mathbf{F}_2) \to R$ is (essentially) an edge homomorphism.
- Use properties of Adams spectral sequences, plus some computations, to show that, mod nilpotence, R detects everything in $\mathrm{Ext}_A^*(\mathbf{F}_2, \mathbf{F}_2)$.

□

Lemma

Vanishing lines in Adams spectral sequences are generic.

That is: For fixed ring spectrum E and fixed slope m, the collection of all objects X for which, at some term of the Adams spectral sequence

$$\mathrm{Ext}^*_{E_*E}(E_*, E_*X) \Longrightarrow \pi_*X,$$

there is a vanishing line of slope m, is a thick subcategory.
To prove the theorem, show that for each $m > 0$, there is a vanishing line of slope m at some term of the spectral sequence

$$\mathrm{Ext}^*_{A//D}(\mathbf{F}_2, \mathrm{Ext}^*_D(\mathbf{F}_2, \mathbf{F}_2)) \Longrightarrow \mathrm{Ext}^*_A(\mathbf{F}_2, \mathbf{F}_2).$$

This implies that everything not on the bottom edge is nilpotent. . .

This Steenrod analogue of Nishida's theorem can be modified to give a nilpotence theorem (à la DHS).

Question

What about a thick subcategory theorem?

Question

What about the Bousfield lattice?

Question

What about a periodicity theorem and other chromatic structure?

Other ideas:

- Study Bousfield localization in Stable(A_*).
- Investigate the telescope conjecture.
- Investigate the odd primary case.
- Investigate analogous categories coming from BP_*BP or other Hopf algebroids.

References

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