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Research description
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My research is in *stable homotopy theory*, which is a subfield of topology, one of the main branches of mathematics. Stable homotopy theory is roughly seventy years old, and much of the research in the field has been focused on one hard problem: compute the stable homotopy groups of spheres. Many world-class mathematicians have worked on this (for example, Fields Medal winners Serre, Thom, Novikov, and Quillen did much of their early work in homotopy theory), and while they couldn't solve this fundamental problem, they each made important contributions to the field. This happens whenever mathematicians come across a hard and interesting problem—even if they don't solve it, they develop useful tools, they make partial progress, and they do research that leads in new directions.

As a consequence, stable homotopy theory is a well-developed field, with powerful tools for making progress on the fundamental problem of the field, and with interesting directions for active research. My research has been driven by the following question: to what extent can one apply those powerful tools, or translate those directions for research, into other fields of mathematics? And more generally, what connections are there between stable homotopy theory and other fields?

Axiomatic stable homotopy theory. For the first few years after my PhD, I did "pure" stable homotopy theory, but I gradually started to understand some of the connections between stable homotopy theory and modular representation theory of finite groups, which is a subfield of algebra – another of the main branches of mathematics. Other mathematicians had made similar connections, and this led Hovey, Strickland, and me to formalize these connections in our book *Axiomatic stable homotopy theory* [7]. This book provides a list of conditions that might apply to various mathematical settings; if some particular setting satisfies all of those conditions, then we show that one can "do stable homotopy theory" in that setting—many of the tools and theorems from stable homotopy theory apply, and the stable homotopy theoretic point of view leads to new directions for studying that setting.

This approach was not entirely new: mathematicians had been doing analogous things informally for some time. Our work set the stage for doing the same but more easily and more formally: rather than saying, "the following is an analogue of a result in stable homotopy theory" and then proceeding with an adapted version of the proof of the original result, one can now frequently say, "since the axioms for a stable homotopy category apply here, we can immediately deduce the following consequences."

This approach is gaining popularity, as a search at the Hopf Topology Archive—an on-line archive of recent papers in topology—shows: see papers by Christensen, Keller, and Neeman; Dwyer and Greenlees; Mahowald and Rezk; and Schwede and Shipley. It is catching on outside topology, as well; for example, recent work of Benson and Gnacadja in algebra (and in particular in modular representation theory) uses the stable homotopy theoretic approach.

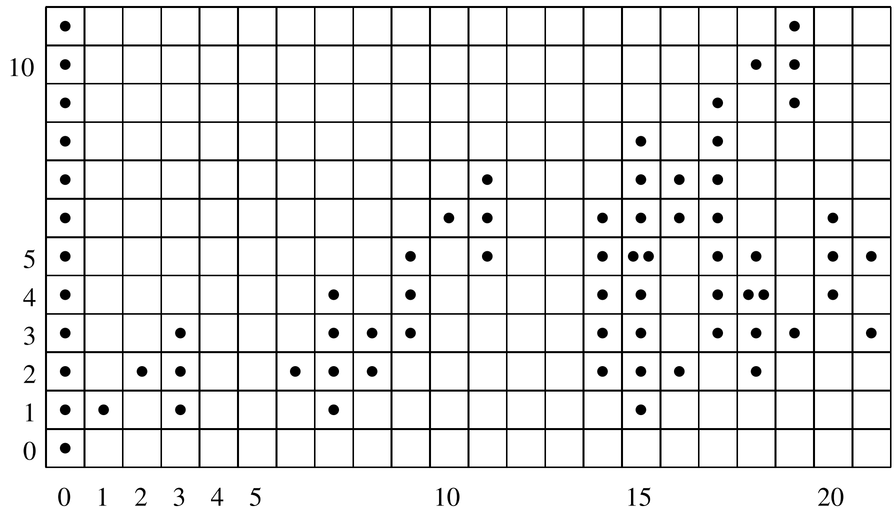
Quillen stratification for the Steenrod algebra. The *Steenrod algebra* provides another connection between homotopy theory and algebra. The Steenrod algebra is a mathematical object introduced around fifty years ago, and studying its algebraic properties often leads to topological insights. In particular, a piece of machinery called the *Adams spectral sequence* allows one to translate information about the Steenrod algebra to information about the stable homotopy groups of spheres; again, understanding these stable homotopy groups is the central problem in stable homotopy theory. Somewhat more precisely, the Adams spectral sequence connects the *cohomology of the Steenrod algebra* to the stable homotopy groups of spheres; one usually views the cohomology of the Steenrod algebra as the input to this piece of machinery, and the stable homotopy groups of spheres as the output. In my paper, "Quillen stratification for the Steenrod algebra" [8], I prove a theorem about the cohomology of the Steenrod algebra; in order to describe this result, I need to provide a bit more background.

Groups are basic objects in the field of algebra. The study of the cohomology of groups is of fairly recent vintage (around fifty years old), but there are many beautiful results in this field. One famous result is Quillen's stratification theorem [10], which allows one to compute cohomology of groups from simpler data.

Now, the Steenrod algebra acts in many ways as if it were a group, and so the cohomology of the Steenrod algebra is closely related to the cohomology of groups. The main result of my paper [8] is a version of Quillen's theorem for the Steenrod algebra. This is important in several ways: it gives computational information about the cohomology of the Steenrod algebra, and hence potential information about the central goal, stable homotopy groups of spheres. It also strengthens the analogies between the study of groups and the study of the Steenrod algebra.

I am going to describe the theorem, to put it into context among other results on the cohomology of the Steenrod algebra. You can picture the cohomology of the Steenrod algebra as follows: imagine that you have a piece of graph paper which has a bottom edge and a left-hand edge but extends infinitely up and to the right. The bottom row is row zero and the leftmost column is column zero. Each box in the grid is either empty or has some number of dots in it, and the pattern of dots represents the cohomology of the Steenrod algebra. The simplest question to ask is, how many dots are in each box? Previous authors have computed pieces of this; for example, rows 0–3 are completely understood. (In row zero, there is only one dot, and that is in column zero. In row one, there is a single dot in columns 0, 1, 3, 7, 15, 31, …, and all the other boxes in this row are empty. Rows two and three are more complicated.) Column zero is also understood: there is a single dot in every row of column zero. Every other column has only finitely many dots in it; in column n, there are no dots above row $\frac{n}{2}+1$ or so. Because of this, the columns are a bit easier to understand, and approximately the first 100 columns have been computed. Most of these computations and results are at least thirty years old.

Here is a picture of the first 21 columns; remember that column zero has a dot in every row, not just the rows pictured here.



There is more structure, though: there are families of dots which lie evenly spaced along lines through box $(0,0)$. (I'll call the box in row i and column j "box (i,j).") The dot in box $(0,0)$ is a member of every family. For example, all of the dots in the first column form a family: there is one dot in box $(n,0)$ for every number $n \geq 0$. Not every dot is in a family; for instance, the dot in box $(1,1)$ is not. There are dots in boxes $(0,0)$, $(1,1)$, $(2,2)$, and $(3,3)$, but box $(4,4)$ is empty, so the pattern breaks down.

My theorem describes *all* of these families; for example, my theorem says that there is a family along the line through the boxes

$$(0,0),\ (2,7),\ (4,14),\ (6,21),\ (8,28),\ldots.$$

The theorem does not give complete information, though: in this case, it doesn't say exactly where those dots are—they might be in every one of these boxes, or every other one, or every third one, etc. (It turns out that this particular family consists of a dot in every other box in that list.) The theorem produces infinitely many different families like this.

This theorem is completely new—it doesn't extend previous box-by-box or column-by-column calculations, but rather finds interesting patterns in the big picture. This is the mark of a good theorem: it addresses a topic of interest to a group of researchers (not just the author), and dramatically extends what is known about that topic.

Quillen's original proof does not generalize to the Steenrod algebra, for certain technical reasons—it only applies to what are called *compact* groups, and the Steenrod algebra fails to satisfy any reasonable analogue of compactness. Thus I used a completely different approach, and that approach is worth discussing. As I mentioned above, the Adams spectral sequence is one particular tool used to study the stable homotopy groups of spheres. Because of the axiomatic approach discussed earlier, whenever you have a setting which satisfies the axioms for stable homotopy theory, there is a version of the Adams spectral sequence, which can be used to study whatever is playing the role of the stable homotopy groups of spheres. Using the Steenrod algebra as a base of operations, Hovey, Strickland and I [7] established a setting in which the axioms apply, and in which the role of the stable homotopy groups of spheres is played by the cohomology of the Steenrod algebra. In other words, in this new setting, the cohomology of the Steenrod algebra, which is the input for the classical Adams spectral sequence, becomes the *output* for this new version of the Adams spectral sequence. This is an unexpected twist: I used a standard tool—the Adams spectral sequence—in a nonstandard way, turning it around to make its input into its output.

Notice, by the way, that the motivation for this work came from using the connection between stable homotopy theory and algebra, but in the other direction: the parallel between the two fields suggested that I should try to model the study of the Steenrod algebra and the Adams spectral sequence on results from group cohomology. This point of view led me to the discovery of the Quillen stratification theorem for the Steenrod algebra.

This paper was published in the *Annals of Mathematics*, which is one of the premier mathematics journals. The ranking of mathematics journals is subjective, of course, and it also varies from field to field, but most algebraic topologists would agree that the *Annals* is the best journal in our field.

Stable homotopy over the Steenrod algebra. As mentioned in the previous section, the axioms for stable homotopy theory apply in a setting centered on the Steenrod algebra, and I used this observation to prove my version of the Quillen stratification theorem. In my research monograph *Stable homotopy over the Steenrod algebra* [9], I engaged in an in-depth exploration of the Steenrod algebra from the stable homotopy theoretic viewpoint. The book starts with a description of a stable homotopy theoretic setting which can be applied to the Steenrod algebra, but also more generally; I also made some easy observations about this setting, but these easy observations simplify some machinery which is quite complicated from the old-fashioned point of view—this demonstrates the power of the stable homotopy theoretic approach. For the rest of the book, I focused on the Steenrod algebra; starting from scratch, I re-developed the work of some other authors in this new setting, and then presented some more recent work of mine, like the Quillen stratification theorem, as well as some new results.

For example, I explored objects called *thick subcategories* in this setting. Hopkins and Smith [4] classified the thick subcategories in ordinary stable homotopy theory (that is, in the original setting which inspired the axioms). Their theorem is deep and important; unfortunately, their proof used some specific computations which only work in ordinary stable homotopy theory, not in other settings. So a natural question is, what are the thick subcategories in the setting of stable homotopy theory over the Steenrod algebra? No one has been able to answer this, but I made some progress. First, the Hopkins-Smith theorem was a consequence of a result of Devinatz, Hopkins, and Smith [3] called the Nilpotence Theorem. One can view my Quillen stratification theorem as a version of this theorem, and I used this parallel to find a conjecture for the classification of thick subcategories for the Steenrod algebra. Next, I provided evidence for this conjecture, and I also proved some small parts of it. Finally, I gave a possible outline for a proof of half of the conjecture. Presenting a conjecture, along with partial progress, is an important aspect of mathematical publication, because it helps to direct future research.

My first goal in writing the book was to carefully describe and analyze the stable homotopy theoretic setting for the Steenrod algebra. Then I used this description and analysis to achieve my other goals: to understand more about the Steenrod algebra and its cohomology—that is, to prove some new results; and to provide an advertisement for the stable homotopy theoretic approach, by giving a well-developed illustration of its use and power. And of course, further research will be easier with the groundwork in place.

Recent joint work with Hovey. Mark Hovey (Wesleyan University) and I have collaborated a number of times over the past five years. In two recent papers, "Stably thick subcategories of modules over Hopf algebras" [6] and "Galois theory of thick subcategories in modular representation theory" [5], we used axiomatic stable homotopy theory to do the following: recast results in group cohomology due to Benson, Carlson, and Rickard [1, 2] in what we view as a nicer way, strengthen those results in the group theory setting, and generalize those results to other settings.

Recall from above that groups are important objects in algebra, and also that the Steenrod algebra has many similarities to a group. Group theorists are interested in the cohomology of groups, just as topologists are interested in the cohomology of the Steenrod algebra. Furthermore, our axiomatic approach to stable homotopy theory works for groups: given a group, there is a setting in which one can "do stable homotopy theory" as a way to study the cohomology of that group. In particular, a good question to ask is, what are the thick subcategories in this setting?

Benson, Carlson, and Rickard [2] were able to answer this question for what are called *finite* groups. The stable homotopy theoretic approach suggested to us different emphases to their work, though; for example, while they would define a term one way and then prove that a second possible definition was equivalent, from our point of view, the second definition is much more natural. For another example, they proved perhaps half a dozen main results in their paper [1]; from our point of view, one of those results was the linchpin—given that one, the others were easy consequences. We used this observation to reproduce their results in a more general setting. In particular, while they worked exclusively with groups, groups have a generalization called *Hopf algebras*, of which the Steenrod algebra is an example. (Actually, they worked with finite groups, which generalize to *finite-dimensional* Hopf algebras, which the Steenrod algebra is *not*, so this work does not directly apply to it, unfortunately.) Groups in general are easier to work with, but by focusing on the linchpin result, we were able to lay the groundwork for extending their results to finite-dimensional Hopf algebras, and we completed this extension for several families of these Hopf algebras. One important such family is related to the Steenrod algebra; the members of the family are essentially "toy versions" of the Steenrod algebra, and so these results do give some indirect information about the Steenrod algebra case.

Their results also required a technical hypothesis—they needed to work with an *algebraically closed field*; in the "Galois theory" paper, we proved that this hypothesis was unnecessary.

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