

ABSOLUTE CONTINUITY FOR RANDOM ITERATED FUNCTION SYSTEMS WITH OVERLAPS

YUVAL PERES, KÁROLY SIMON, AND BORIS SOLOMYAK

ABSTRACT. We consider linear iterated function systems with a random multiplicative error on the real line. Our system is $\{x \mapsto d_i + \lambda_i Y x\}_{i=1}^m$, where $d_i \in \mathbb{R}$ and $\lambda_i > 0$ are fixed and $Y > 0$ is a random variable with an absolutely continuous distribution. The iterated maps are applied randomly according to a stationary ergodic process, with the sequence of i.i.d. errors y_1, y_2, \dots , distributed as Y , independent of everything else. Let h be the entropy of the process, and let $\chi = \mathbb{E}[\log(\lambda Y)]$ be the Lyapunov exponent. Assuming that $\chi < 0$, we obtain a family of conditional measures $\nu_{\mathbf{y}}$ on the line, parametrized by $\mathbf{y} = (y_1, y_2, \dots)$, the sequence of errors. Our main result is that if $h > |\chi|$, then $\nu_{\mathbf{y}}$ is absolutely continuous with respect to the Lebesgue measure for a.e. \mathbf{y} . We also prove that if $h < |\chi|$, then the measure $\nu_{\mathbf{y}}$ is singular and has dimension $h/|\chi|$ for a.e. \mathbf{y} . These results are applied to a randomly perturbed IFS suggested by Y. Sinai, and to a class of random sets considered by R. Arratia, motivated by probabilistic number theory.

1. INTRODUCTION

Let $\{f_1, \dots, f_m\}$ be an iterated function system (IFS) on the real line, where the maps are applied according to the probabilities (p_1, \dots, p_m) , with the choice of the map random and independent at each step. We assume that the system is contracting on average, that is, the Lyapunov exponent χ (appropriately defined) is negative. In this paper the maps will be linear, $f_i(x) = \lambda_i x + d_i$, and then $\chi := \sum_{i=1}^m p_i \log \lambda_i$. If $\chi < 0$, then there is a well-defined invariant probability

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measure ν on \mathbb{R} (see [3]). It is of interest to determine whether this measure is singular or absolutely continuous, and if it is singular, to compute its Hausdorff dimension

$$\dim_H(\nu) = \inf\{\dim_H(Y) : \nu(\mathbb{R} \setminus Y) = 0\}.$$

Let $h = -\sum_{i=1}^m p_i \log p_i$ be the entropy of the underlying Bernoulli process. It was proved in [14] for non-linear contracting on average IFS (and later extended in [6]) that

$$\dim_H(\nu) \leq h/|\chi|.$$

A question arises what happens when the entropy is greater than the absolute value of the Lyapunov exponent. One can expect that, at least “typically,” the measure ν is absolutely continuous when $h/|\chi| > 1$. Such results are known for contracting IFS (see [15, 16, 12, 13]), but extending them to the case when it is only contracting on average remains a challenge.

In this paper we study a modification of the problem which makes it more tractable, namely we consider linear IFS with a random multiplicative error. Our system is $\{x \mapsto d_i + \lambda_i Y x\}_{i=1}^m$, where $d_i \in \mathbb{R}$ and $\lambda_i > 0$ are fixed and $Y \geq 0$ is a random variable with an absolutely continuous distribution. The iterated maps are applied randomly according to a stationary ergodic process, with the sequence of i.i.d. errors y_1, y_2, \dots , distributed as Y , independent of everything else. Let h be the entropy of the process, and let $\chi = \mathbb{E}[\log(\lambda Y)]$ be the Lyapunov exponent (the symbol \mathbb{E} denotes expectation). Assuming that $\chi < 0$, we obtain a family of conditional measures $\nu_{\mathbf{y}}$ on the line, parametrized by $\mathbf{y} = (y_1, y_2, \dots)$, the sequence of errors. Our main result is that if $h > |\chi|$, then $\nu_{\mathbf{y}}$ is absolutely continuous with respect to the Lebesgue measure \mathcal{L} for a.e. \mathbf{y} . We also prove that if $h < |\chi|$, then the measure $\nu_{\mathbf{y}}$ is singular and has dimension $h/|\chi|$ for a.e. \mathbf{y} . Random IFS are quite well understood under separation conditions (see [9, 10]), and the novelty here is that overlaps are allowed.

Before stating the general results precisely, we describe two examples that motivated our work. Consider the random series

$$X = 1 + Z_1 + Z_1 Z_2 + \dots + Z_1 Z_2 \cdots Z_n + \dots \tag{1.1}$$

where Z_i are i.i.d. random variables, taking the values in $\{1 - a, 1 + a\}$ with probabilities $(\frac{1}{2}, \frac{1}{2})$, for a fixed parameter $a \in (0, 1)$. The series converges almost surely, since $\chi = \mathbb{E}[\log Z] = \frac{1}{2} \log(1 - a^2) < 0$. Let ν^a denote the distribution of X . Sinai

[personal communication], motivated by a statistical analog of the well-known open “ $3n + 1$ problem,” asked for which a the distribution of X is absolutely continuous. Observe that ν^a is supported on $[\frac{1}{a}, \infty)$ for all $a > 0$. Note that ν^a is the invariant measure for the IFS $\{1 + (1 - a)x, 1 + (1 + a)x\}$, with probabilities $(\frac{1}{2}, \frac{1}{2})$. If $h < |\chi|$, that is, $\log 2 < -\frac{1}{2} \log(1 - a^2)$, or $a > \sqrt{3}/2$, then the measure is singular. It is natural to predict that $\nu^a \ll \mathcal{L}$ for a.e. $a \in (0, \sqrt{3}/2)$. While this question remains open, here we solve a “randomly perturbed version.”

Proposition 1.1. *Consider the random sum (1.1), with $Z_i = \lambda_i Y$, where $\lambda_i \in \{1 - a, 1 + a\}$ with probabilities $(\frac{1}{2}, \frac{1}{2})$ and Y has an absolutely continuous distribution on $(1 - \varepsilon_1, 1 + \varepsilon_2)$ for small ε_1 and ε_2 , with a bounded density, such that $\mathbb{E}[\log Y] = 0$. The “errors” y_i at each step are i.i.d. with the distribution of Y , and are independent of everything else. Let $\nu_{\mathbf{y}}^a$ be the conditional distribution of Z , given a sequence of errors $\mathbf{y} = (y_1, y_2, \dots)$.*

(a) *If $a \in (0, \sqrt{3}/2)$, then $\nu_{\mathbf{y}}^a \ll \mathcal{L}$ for a.e. \mathbf{y} ;*

(b) *if $a \geq \sqrt{3}/2$, then $\nu_{\mathbf{y}}^a \perp \mathcal{L}$ and $\dim_H(\nu_{\mathbf{y}}^a) = 2 \log 2 / \log \frac{1}{1 - a^2}$ for a.e. \mathbf{y} .*

Our second example is probabilistic in its origin (rather than a random perturbation of a deterministic one, as above). It comes from a question of Arratia (see Section 22 in [2]), who considered the following distributions, motivated by some questions in probabilistic number theory. Let $Y = U^{1/\theta}$, where U has the uniform distribution on $[0, 1]$, and consider $X_i = Y_1 \cdots Y_i$, where Y_i are i.i.d. with the distribution of Y . The process $\{X_i\}$ is known as the scale-invariant Poisson process with intensity $\theta x^{-1} dx$. Consider the random sum $Z = \sum_{i \geq 1} J_i X_i$ where J_1, J_2, \dots are “fair coins” with values in $\{0, 1\}$, independent of each other and of everything else. One is interested in the conditional distribution $\nu_{\mathbf{y}}^\theta$ of Z given the process $\{Y_i\}$, and in its support, $S_{\mathbf{y}}^\theta = \{\sum_{i=1}^\infty a_i X_i : a_i \in \{0, 1\}\}$. Observe that this fits into our class of IFS $\{d_i + \lambda_i Y x\}_{i=1}^m$, by taking $m = 2$, $d_1 = 0, d_2 = 1, \lambda_1 = \lambda_2 = 1$. The distribution of $U^{1/\theta}$ has the density $\theta x^{\theta-1} \mathbf{1}_{[0,1]}$, so we have $\chi = \mathbb{E}[\log \lambda Y] = \mathbb{E}[\log Y] = -\theta^{-1}$. The entropy of the “fair coins” process is $h = \log 2$, so $h/|\chi| = \theta \log 2$.

Proposition 1.2. *Let $Z = \sum_{i \geq 1} J_i X_i$ as above. Let $\nu_{\mathbf{y}}^\theta$ be the conditional distribution of Z given the process $\{Y_i\}$, and let $S_{\mathbf{y}}^\theta$ be the support of $\nu_{\mathbf{y}}^\theta$.*

(a) *If $\theta > 1/\log 2$, then $\nu_{\mathbf{y}}^\theta \ll \mathcal{L}$ with a density in $L^2(\mathbb{R})$, hence $\mathcal{L}(S_{\mathbf{y}}^\theta) > 0$, for a.e. \mathbf{y} .*

(b) If $\theta > 2/\log 2$, then $\nu_{\mathbf{y}}^\theta \ll \mathcal{L}$ with a continuous density, hence $S_{\mathbf{y}}^\theta$ contains an interval, for a.e. \mathbf{y} .

(c) If $\theta \in (0, 1/\log 2]$, then $\mathcal{L}(S_{\mathbf{y}}^\theta) = 0$, and $\dim_H(S_{\mathbf{y}}^\theta) = \dim_H(\nu_{\mathbf{y}}^\theta) = \theta \log 2$ for a.e. \mathbf{y} .

An intriguing open problem is whether $S_{\mathbf{y}}^\theta$ contains intervals for a.e. \mathbf{y} when $\theta \in (\frac{1}{\log 2}, \frac{2}{\log 2})$. The proof of Proposition 1.2 is easily adapted to the Poisson-Dirichlet distributions where the variables $X_i = Y_1 \cdots Y_i$ are replaced by $\tilde{X}_i = Y_1 \cdots Y_{i-1}(1 - Y_i)$ ordered by size, see the equations (6.2) and (8.2) in [2].

Our results yield many variants of Proposition 1.2. For example, let $\tilde{S}_{\mathbf{y}}^\theta = \{\sum_{i=1}^\infty a_i X_i : a_i \in \{0, 1\}, a_i a_{i+1} = 0, i \geq 1\}$; in other words, we consider only the sums corresponding to the ‘‘Fibonacci’’ shift of finite type. Let $\tau = (1 + \sqrt{5})/2$.

Proposition 1.3. (a) If $\theta > 1/\log \tau$, then $\mathcal{L}(\tilde{S}_{\mathbf{y}}^\theta) > 0$, for a.e. \mathbf{y} .

(b) If $\theta \in (0, 1/\log \tau)$, then $\mathcal{L}(\tilde{S}_{\mathbf{y}}^\theta) = 0$ and $\dim_H(\tilde{S}_{\mathbf{y}}^\theta) = \theta \log \tau$ for a.e. \mathbf{y} .

2. STATEMENT OF RESULTS

Consider a random variable Y with an absolutely continuous distribution η on $(0, \infty)$, such that for some $C_1 > 0$ we have

$$\frac{d\eta}{dx} \leq C_1 x^{-1}, \quad \forall x > 0. \quad (2.1)$$

Let $\mathbb{R}^{\mathbb{N}}$ be the infinite product equipped with the product measure $\eta_\infty := \eta^{\mathbb{N}}$. Let μ be an ergodic σ -invariant measure on $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$, where σ is the left shift. Denote by $h(\mu)$ the entropy of the measure μ . We consider linear IFS with a random multiplicative error $\{x \mapsto d_i + \lambda_i Y x\}_{i=1}^m$, where $d_i \in \mathbb{R}$ and $\lambda_i > 0$ are fixed. The iterated maps are applied randomly according to the stationary measure μ , with the sequence of i.i.d. errors y_1, y_2, \dots , distributed as Y , independent of the choice of the function. The Lyapunov exponent of the IFS is defined by

$$\chi(\mu, \eta) := \mathbb{E}[\log(\lambda Y)] = \mathbb{E}[\log Y] + \int_{\Sigma} \log(\lambda_{i_1}) d\mu(\mathbf{i}).$$

Throughout the paper, we assume that

$$\chi(\mu, \eta) < 0, \quad (2.2)$$

which means that the IFS is contracting on average. The natural projection $\Pi : \Sigma \times \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}$ is defined by

$$\Pi(\mathbf{i}, \mathbf{y}) := d_{i_1} + d_{i_2} \lambda_{i_1} y_1 + \cdots + d_{i_{n+1}} \lambda_{i_1 \dots i_n} y_{1 \dots n} + \cdots, \quad (2.3)$$

where $\mathbf{i} = (i_1, i_2, \dots)$, $\lambda_{i_1 \dots i_n} := \lambda_{i_1} \cdots \lambda_{i_n}$, and $y_{1 \dots n} = y_1 \cdots y_n$. Note that $\Pi(\mathbf{i}, \mathbf{y})$ is a Borel map defined $\mu \times \eta_\infty$ a.e., since $n^{-1} \log(\lambda_{1 \dots n} y_{1 \dots n}) \rightarrow \mathbb{E}[\log(\lambda Y)] < 0$ a.e., by the Birkhoff Ergodic Theorem. For a fixed $\mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ we define $\Pi_{\mathbf{y}} : \Sigma \rightarrow \mathbb{R}$ and the measure $\nu_{\mathbf{y}}$ on \mathbb{R} as

$$\Pi_{\mathbf{y}}(\mathbf{i}) := \Pi(\mathbf{i}, \mathbf{y}) \quad \text{and} \quad \nu_{\mathbf{y}} := (\Pi_{\mathbf{y}})_* \mu. \quad (2.4)$$

We need to impose a condition which guarantees that the maps of the IFS are sufficiently different. We consider two cases which cover the interesting examples that we know of. We assume that either all the digits are distinct:

$$d_i \neq d_j, \quad \text{for all } i \neq j, \quad (2.5)$$

or all the digits d_i are equal to some $d \neq 0$ (which we can assume to be 1, without loss of generality), but the average contraction ratios are all distinct:

$$d_i = 1, \quad \lambda_i \neq \lambda_j, \quad \text{for all } i \neq j. \quad (2.6)$$

Theorem 2.1. *Let $\nu_{\mathbf{y}}$ be the conditional distribution of the sum (2.3) given $\mathbf{y} = (y_1, y_2, \dots)$. We assume that (2.1), (2.2) hold, and either (2.5) or (2.6) is satisfied.*

(a) *If $h(\mu) > |\chi(\mu, \eta)|$, then*

$$\nu_{\mathbf{y}} \ll \mathcal{L} \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y}. \quad (2.7)$$

(b) *If $h(\mu) \leq |\chi(\mu, \eta)|$, then*

$$\dim_H(\nu_{\mathbf{y}}) = \frac{h(\mu)}{|\chi(\mu, \eta)|} \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y}. \quad (2.8)$$

Assuming that μ is a product (Bernoulli) measure, that is, $\mu = (p_1, \dots, p_m)^{\mathbb{N}}$ and $h(\mu) = |\chi(\mu, \eta)|$, we can show that the measure $\nu_{\mathbf{y}}$ is singular for a.e. \mathbf{y} .

Proposition 2.2. *Suppose that $\mu = (p_1, \dots, p_m)^{\mathbb{N}}$ and $Y > 0$ is any random variable, such that (2.2) holds and $(\mathbf{i}, \mathbf{y}) \mapsto p_{i_1} \lambda_{i_1}^{-1} y_1^{-1}$ is non-constant on $\Sigma \times \mathbb{R}^{\mathbb{N}}$. If $h(\mu) = -\chi(\mu, \eta)$, then $\nu_{\mathbf{y}} \perp \mathcal{L}$ for η_∞ a.e. \mathbf{y} .*

Note that in the proposition we do not make any assumptions on the distribution of Y . If Y is any non-constant random variable, then the proposition applies. On the other hand, it includes the case when Y is constant (in other words, this is a usual IFS with no randomness), but λ_j/p_j is not constant. Then of course Y can be eliminated altogether and there is no a.e. \mathbf{y} in the statement. In the special case $Y \equiv 1$ and $\lambda_i < 1$ for all $i \leq m$, our statement is contained in [13, Th.1.1(ii)].

We should emphasize that Proposition 2.2, as well as the upper dimension estimate in (2.8), are rather standard; they are included for completeness, in order to indicate that our results are sharp.

Next we discuss two special cases which include the examples from the Introduction.

2.1. Sums of products of i.i.d. random variables. Suppose that (2.6) holds and $\mu = (p_1, \dots, p_m)^{\mathbb{N}}$. Then we are led to a random variable

$$X = 1 + Z_1 + Z_1 Z_2 + \dots + Z_1 Z_2 \cdots Z_n + \dots \quad (2.9)$$

where Z_i are independent with the distribution of $Z = \lambda Y$, where $\lambda = \lambda_j$ with probability p_j and Y , independent of λ , satisfies (2.1), as in the general case. Then $\chi(\mu, \eta) = \sum_{i=1}^m p_i \log \lambda_i + \mathbb{E}[\log Y]$ and $h(\mu) = -\sum_{i=1}^m p_i \log p_i$. The measure $\nu_{\mathbf{y}}$ is the conditional distribution of X given $\mathbf{y} = (y_1, y_2, \dots)$, a realization of the process $\{Y_i\}$.

Thus Proposition 1.1 on the randomly perturbed Sinai's problem is a special case of Theorem 2.1 and Proposition 2.2.

2.2. Homogeneous case: random measures. Suppose that $\lambda_i = \lambda$ for all $i \leq m$, so we have $d_i \neq d_j$ for $i \neq j$ by (2.5). We are led to the random sums

$$X = \sum_{i=1}^{\infty} J_i Y_1 \cdots Y_{i-1} \lambda^{i-1}, \quad (2.10)$$

where Y_i are i.i.d. with the absolutely continuous distribution η , and J_i take the values in $\{d_1, \dots, d_m\}$, are independent of $\{Y_j\}$, and are chosen according to an ergodic σ -invariant measure μ . Then Theorem 2.1 applies to $\nu_{\mathbf{y}}$, the conditional distribution of X given $\mathbf{y} = (y_1, y_2, \dots)$, a realization of the process $\{Y_j\}$. The Lyapunov exponent $\chi(\eta) = \log \lambda + \mathbb{E}[\log Y]$ does not depend on μ .

When μ is Bernoulli, that is, $\mu = (p_1, \dots, p_m)^{\mathbb{N}}$, there is an alternative method to study $\nu_{\mathbf{y}}$ which goes back to the work of Kahane and Salem [7] and uses Fourier transform. It requires a stronger assumption on the distribution η , namely that

$$\eta \text{ has compact support and } d\eta/dx \text{ is of bounded variation.} \quad (2.11)$$

Theorem 2.3. *Let $\nu_{\mathbf{y}}$ be the conditional distribution of the sum (2.3) given $\mathbf{y} = (y_1, y_2, \dots)$, defined by (2.4). We assume that (2.11) and (2.5) are satisfied, $\mu = (p_1, \dots, p_m)^{\mathbb{N}}$ and $\lambda_i = \lambda$ for all $i \leq m$. Suppose that $\chi(\eta) = \log \lambda + \mathbb{E}[\log Y] < 0$.*

- (a) If $|\log(\sum_{i=1}^m p_i^2)| > |\chi(\eta)|$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$ with a density in $L^2(\mathbb{R})$ for η_∞ a.e. \mathbf{y} .
- (b) If $|\log(\sum_{i=1}^m p_i^2)| > 2|\chi(\eta)|$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$ with a continuous density for η_∞ a.e. \mathbf{y} .

Observe that in the uniform case, when $p_i = \frac{1}{m}$ for $i \leq m$, we get the same threshold $|\chi(\eta)| = \log m$ for absolute continuity in Theorem 2.1(a) and for absolute continuity with a density in L^2 , in Theorem 2.3(a).

2.3. Homogeneous case: random sets. The results on random measures yield information on random sets. Recall that $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$, and let $\Gamma \subset \Sigma$ be a closed σ -invariant subset. For a digit set $\{d_1, \dots, d_m\}$ and $\mathbf{y} \in (0, \infty)^{\mathbb{N}}$ consider

$$S_\Gamma(\mathbf{y}) = \left\{ \sum_{i=1}^{\infty} d_{a_i} y_{1\dots(i-1)} : \{a_i\}_1^\infty \in \Gamma \right\}.$$

We let $S(\mathbf{y}) = S_\Sigma(\mathbf{y})$. Denote by $h_{\text{top}}(\Gamma)$ the topological entropy of (Γ, σ) . In the next three corollaries we consider a digit set $\{d_1, \dots, d_m\}$ satisfying (2.5), that is, all the digits are assumed to be distinct. For a random variable η we let $\chi(\eta) := \int \log \eta d\eta < 0$.

Corollary 2.4. *Suppose that η satisfies (2.1) and $\chi(\eta) < 0$.*

- (a) *If $h_{\text{top}}(\Gamma) > |\chi(\eta)|$, then $\mathcal{L}(S_\Gamma(\mathbf{y})) > 0$ for η_∞ a.e. \mathbf{y} .*
- (b) *If $h_{\text{top}}(\Gamma) \leq |\chi(\eta)|$, then $\dim_H(S_\Gamma(\mathbf{y})) = h_{\text{top}}(\Gamma)/|\chi(\eta)|$ for η_∞ a.e. \mathbf{y} .*

Corollary 2.5. *Suppose that η satisfies (2.11) and $\chi(\eta) < 0$. We consider $\Gamma = \Sigma$.*

- (a) *If $\log m > 2|\chi(\eta)|$, then $S(\mathbf{y})$ contains an interval for η_∞ a.e. \mathbf{y} .*
- (b) *If $\log m \leq |\chi(\eta)|$, then $\dim_H(S(\mathbf{y})) = \log m/|\chi(\eta)|$ for η_∞ a.e. \mathbf{y} .*

Corollary 2.6. *Suppose that Y is any non-constant random variable on $(0, \infty)$ such that $\chi(\eta) < 0$. If $\log m = |\chi(\eta)|$, then $\mathcal{L}(S(\mathbf{y})) = 0$ for η_∞ a.e. \mathbf{y} .*

The results of this subsection imply the statements on Arratia's question and its variants from the Introduction. More precisely, parts (a) and (b) of Proposition 1.2 follow from Theorem 2.3. Part (c) of Proposition 1.2 follows from Theorem 2.1(b) and Corollary 2.6. Proposition 1.3 follows from Corollary 2.4, since $\Gamma = \{(a_i)_1^\infty \in \{0, 1\}^{\mathbb{N}} : a_i a_{i+1} = 0, i \geq 1\}$ has topological entropy $\log \frac{1+\sqrt{5}}{2}$.

The rest of the paper is organized as follows. Theorem 2.1(a) is proved in Sections 3 and 4; the latter also contains a key ‘‘transversality lemma,’’ which is

used in the proof of both Theorem 2.1(a) and the lower estimate in Theorem 2.1(b). Then Theorem 2.1(b) is derived in Section 5. Section 6 is devoted to the proofs of other results, especially Theorem 2.3, following the method of Kahane and Salem. Finally, Section 7 contains some open questions.

3. PRELIMINARIES AND THE PROOF OF THEOREM 2.1(A)

Notation. For $\omega \in \{1, \dots, m\}^n$ we denote by $[\omega]$ the cylinder set of $\mathbf{i} \in \Sigma$ which start with ω . For $\mathbf{i} \in \Sigma$ let $[\mathbf{i}, n] = [i_1 \dots i_n]$. For $\mathbf{i}, \mathbf{j} \in \Sigma$ we denote by $\mathbf{i} \wedge \mathbf{j}$ their common initial segment.

By adding the constant $\mathbb{E}[\log Y]$ to $\log \lambda$ and subtracting it from $\log Y$, we can assume without loss of generality that $\mathbb{E}[\log Y] = 0$, so that $\chi(\mu, \eta) = \chi(\mu) = \mathbb{E}[\log \lambda]$. In order to prove Theorem 2.1, we need to make a certain “truncation” both in $\mathbb{R}^{\mathbb{N}}$ and in Σ . By the Law of Large Numbers,

$$n^{-1} \log(y_{1\dots n}) \rightarrow 0 \quad \text{for } \eta_{\infty} \text{ a.e. } \mathbf{y}. \quad (3.1)$$

By Egorov’s Theorem, for any $\varepsilon > 0$ there exists $F_{\varepsilon} \subset \mathbb{R}^{\mathbb{N}}$, with $\eta_{\infty}(F_{\varepsilon}) > 1 - \varepsilon$, such that $(y_{1\dots n})^{1/n} \rightarrow 1$ uniformly on F_{ε} .

Next we do the truncation in Σ . By the Shannon-McMillan-Breiman Theorem,

$$n^{-1} \log(\mu[\mathbf{i}, n]) \rightarrow -h(\mu) \quad \text{for } \mu \text{ a.e. } \mathbf{i} \in \Sigma. \quad (3.2)$$

By the Birkhoff Ergodic Theorem,

$$n^{-1} \log(\lambda_{i_1 \dots i_n}) \rightarrow \chi(\mu) \quad \text{for } \mu \text{ a.e. } \mathbf{i} \in \Sigma. \quad (3.3)$$

Applying Egorov’s Theorem, we can find $G_{\varepsilon} \subset \Sigma$, with $\mu(G_{\varepsilon}) > 1 - \varepsilon$, such that the convergence in (3.2) and (3.3) is uniform on G_{ε} .

Define $\mu_{\varepsilon} = \mu|_{G_{\varepsilon}}$ and let $\nu_{\mathbf{y}}^{\varepsilon} = (\Pi_{\mathbf{y}})_{*} \mu_{\varepsilon}$. We can work with measures $\nu_{\mathbf{y}}^{\varepsilon}$ instead of $\nu_{\mathbf{y}}$. Indeed, if $\nu_{\mathbf{y}}^{\varepsilon} \ll \mathcal{L}$ for all $\varepsilon > 0$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$, and $\dim_H(\nu_{\mathbf{y}}) = \sup_{\varepsilon > 0} \dim_H(\nu_{\mathbf{y}}^{\varepsilon})$. Since we can obviously assume that $F_{\varepsilon} \subset F_{\varepsilon'}$ for $\varepsilon' < \varepsilon$, (2.7) will follow if we prove that

$$\forall \varepsilon > 0, \nu_{\mathbf{y}}^{\varepsilon} \ll \mathcal{L} \quad \text{for } \eta_{\infty} \text{ a.e. } \mathbf{y} \in F_{\varepsilon}. \quad (3.4)$$

Similarly, (2.8) will follow if we prove that

$$\forall \varepsilon > 0, \dim_H(\nu_{\mathbf{y}}^{\varepsilon}) = \frac{h(\mu)}{|\chi(\mu)|} \quad \text{for } \eta_{\infty} \text{ a.e. } \mathbf{y} \in F_{\varepsilon}. \quad (3.5)$$

Beginning of the Proof of Theorem 2.1(a). Fix $\varepsilon \in (0, 1)$; our goal is to prove (3.4) assuming that $-h(\mu) < \chi(\mu) < 0$. We can fix positive $\theta < \rho$ such that

$$-h(\mu) < \log \theta < \log \rho < \chi(\mu) < 0. \quad (3.6)$$

Next fix $\delta > 0$ such that

$$(1 + \delta)\theta < \rho. \quad (3.7)$$

Using the uniform convergence on F_ε and G_ε , we can find $N = N(\varepsilon, \delta, \theta, \rho) \in \mathbb{N}$ such that, in view of (3.1),

$$(1 + \delta)^{-n} \leq y_1 \cdots y_n \leq (1 + \delta)^n \quad \text{for all } n \geq N, \mathbf{y} \in F_\varepsilon, \quad (3.8)$$

and, in view of (3.6),

$$\mu[\mathbf{i}, n] < \theta^n < \rho^n < \lambda_{i_1 \dots i_n}, \quad \text{for all } n \geq N, \mathbf{i} \in G_\varepsilon. \quad (3.9)$$

We can decompose the measure into the sum of measures on cylinders:

$$\nu_{\mathbf{y}}^\varepsilon = \sum_{|\omega|=N} \nu_{\mathbf{y}, \omega}^\varepsilon, \quad \text{where } \nu_{\mathbf{y}, \omega}^\varepsilon := (\mu|_{[\omega] \cap G_\varepsilon}) \circ \Pi_{\mathbf{y}}^{-1}. \quad (3.10)$$

Thus it is enough to show that

$$\nu_{\mathbf{y}, \omega}^\varepsilon \ll \mathcal{L} \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y} \in F_\varepsilon, \forall \omega, |\omega| = N. \quad (3.11)$$

Let $\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) := |\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})|$ and

$$g_r(\mathbf{i}, \mathbf{j}) := \eta_\infty\{\mathbf{y} \in F_\varepsilon : \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r\}. \quad (3.12)$$

Let $P := G_\varepsilon \times G_\varepsilon$ and $\mu_2 := \mu_\varepsilon \times \mu_\varepsilon$. Denote $P^N := \{(\mathbf{i}, \mathbf{j}) \in P : |\mathbf{i} \wedge \mathbf{j}| \geq N\}$.

Proposition 3.1. *There exists $C = C(\varepsilon) > 0$, such that for all $r > 0$,*

$$A(r) := \iint_{P^N} g_r(\mathbf{i}, \mathbf{j}) d\mu_2(\mathbf{i}, \mathbf{j}) \leq Cr. \quad (3.13)$$

We will prove the proposition in the next section. Before that, using the proposition we prove Theorem 2.1(a).

Conclusion of the Proof of Theorem 2.1(a). In order to prove (3.11), it is enough to verify that

$$\mathcal{I} := \sum_{|\omega|=N} \int_{\mathbf{F}_\varepsilon} \int_{\mathbb{R}} \underline{D}(\nu_{\mathbf{y}, \omega}^\varepsilon, x) d\nu_{\mathbf{y}, \omega}^\varepsilon(x) d\eta_\infty(\mathbf{y}) < \infty,$$

where

$$\underline{D}(\nu, x) := \liminf_{r \rightarrow 0} \frac{\nu([x - r, x + r])}{2r},$$

is the lower derivative of a measure ν , see [11, 2.12]. Observe that

$$\sum_{|\omega|=N} \int_{\mathbb{R}} \nu_{\mathbf{y},\omega}^\varepsilon([x-r, x+r]) d\nu_{\mathbf{y},\omega}^\varepsilon(x) = \iint_{P^N} \mathbb{1}_{\{(\mathbf{i},\mathbf{j}): |\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})| \leq r\}} d\mu_2(\mathbf{i}, \mathbf{j})$$

by the definition of $\nu_{\mathbf{y},\omega}^\varepsilon$. Using this with Fatou's Lemma, and exchanging the order of integration, we obtain that

$$\mathcal{I} \leq \liminf_{r \rightarrow 0} (2r)^{-1} A(r), \quad (3.14)$$

where $A(r)$ was defined in (3.13). Thus $\mathcal{I} < \infty$ follows immediately from Proposition 3.1. \square

4. TRANSVERSALITY LEMMA AND THE PROOF OF PROPOSITION 3.1

We begin with a technical lemma, which is a key for the proof of both parts of Theorem 2.1. We are assuming all the conditions of Theorem 2.1, in particular, that either (2.5) or (2.6) holds.

By the definition of φ and Π we have

$$\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) = |d_{i_1} - d_{j_1} + y_1 \Phi(\mathbf{y}, \mathbf{i}, \mathbf{j})|, \quad (4.1)$$

where

$$\Phi(\mathbf{y}, \mathbf{i}, \mathbf{j}) = \lambda_{i_1} d_{i_2} - \lambda_{j_1} d_{j_2} + \sum_{\ell=2}^{\infty} y_{2 \dots \ell} (\lambda_{i_1 \dots i_\ell} d_{i_{\ell+1}} - \lambda_{j_1 \dots j_\ell} d_{j_{\ell+1}}). \quad (4.2)$$

Note that $\Phi(\mathbf{y}, \mathbf{i}, \mathbf{j})$ does not depend on y_1 . If $|\mathbf{i} \wedge \mathbf{j}| = k \geq 1$ then

$$\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) = \lambda_{i_1 \dots i_k} \cdot y_{1 \dots k} \cdot \varphi(\sigma^k \mathbf{y}, \sigma^k \mathbf{i}, \sigma^k \mathbf{j}). \quad (4.3)$$

Lemma 4.1. *Let $\delta > 0$, $\rho \in (0, 1)$, and $N \in \mathbb{N}$. Consider*

$$F = \{\mathbf{y} \in \mathbb{R}^{\mathbb{N}} : y_{1 \dots n} \geq (1 + \delta)^{-n}, \forall n \geq N\},$$

$$G = \{\mathbf{i} \in \Sigma : \lambda_{i_1 \dots i_n} \geq \rho^n, \forall n \geq N\}.$$

There exists $C_2 > 0$ such that for all $k \geq N$, for all $\mathbf{i}, \mathbf{j} \in G$, with $|\mathbf{i} \wedge \mathbf{j}| = k$,

$$\eta_\infty \{\mathbf{y} \in F : \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r\} \leq C_2 (1 + \delta)^k \rho^{-k} r \quad \text{for all } r > 0. \quad (4.4)$$

Proof. First suppose that the condition (2.5) holds. Then $b := \min_{\ell \neq s} |d_\ell - d_s| > 0$. Since $\mathbf{i} \in G$ and $k \geq N$, we have for $\mathbf{y} \in F$ by (4.1), (4.2) and (4.3):

$$\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r \Rightarrow |d_{i_{k+1}} - d_{j_{k+1}} + y_{k+1} \Phi| < (1 + \delta)^k \rho^{-k} r, \quad (4.5)$$

where $\Phi = \Phi(\sigma^k \mathbf{y}, \sigma^k \mathbf{i}, \sigma^k \mathbf{j})$ does not depend on y_{k+1} . Denote $\Delta_{k+1} := d_{i_{k+1}} - d_{j_{k+1}}$; we have $|\Delta_{k+1}| \geq b$ since $i_{k+1} \neq j_{k+1}$. We can assume that $\Delta_{k+1} < 0$; otherwise, we just switch \mathbf{i} and \mathbf{j} . Since the left-hand side of (4.4) is always bounded above by one, (4.4) holds for $r \geq (1+\delta)^{-k} \rho^k b/2$ with the constant $C_2 = 2/b$. If $r < (1+\delta)^{-k} \rho^k b/2$ then

$$\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r \Rightarrow |\Delta_{k+1} + y_{k+1}\Phi| < (1+\delta)^k \rho^{-k} r < b/2, \quad (4.6)$$

and this implies $\Phi > 0$, in view of y_{k+1} being positive and the fact that $\Delta_{k+1} \leq -b$. Moreover, the right-hand side of (4.6) implies

$$y_{k+1} \in B, \quad \text{where } B := \left[\frac{-\Delta_{k+1} - (1+\delta)^k \rho^{-k} r}{\Phi}, \frac{-\Delta_{k+1} + (1+\delta)^k \rho^{-k} r}{\Phi} \right].$$

Note that B depends on y_{k+2}, y_{k+3}, \dots but not on y_{k+1} . We have $(1+\delta)^k \rho^{-k} r < b/2 \leq -\Delta_{k+1}/2$, so

$$B \subset [-\Delta_{k+1}/(2\Phi), \infty) \subset [b/(2\Phi), \infty).$$

By (2.1), we obtain that for any y_{k+2}, y_{k+3}, \dots ,

$$\eta\{y_{k+1} \in B\} \leq C_1(2\Phi/b)\mathcal{L}(B) = C_1(4/b)(1+\delta)^k \rho^{-k} r.$$

This implies the desired inequality (4.4) by Fubini Theorem, since y_{k+1} is independent of y_{k+2}, y_{k+3}, \dots

Now suppose that the condition (2.6) holds. Then by (4.1) and (4.2),

$$\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) = \lambda_{i_1 \dots i_k} \cdot y_{1 \dots k, (k+1)} \cdot |\lambda_{i_{k+1}} - \lambda_{j_{k+1}} + y_{k+2}\Psi|,$$

where

$$\Psi = \lambda_{i_{k+1} i_{k+2}} - \lambda_{j_{k+1} j_{k+2}} + \sum_{\ell=3}^{\infty} y_{(k+3) \dots (k+\ell)} (\lambda_{i_{k+1} \dots i_{k+\ell}} - \lambda_{j_{k+1} \dots j_{k+\ell}})$$

does not depend on y_{k+2} . Since $\mathbf{i} \in G$ and $k \geq N$, we have for $\mathbf{y} \in F$:

$$\varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r \Rightarrow |\lambda_{i_{k+1}} - \lambda_{j_{k+1}} + y_{k+2}\Psi| < (1+\delta)^{k+1} \rho^{-k} r.$$

Here $|\lambda_{i_{k+1}} - \lambda_{j_{k+1}}| \geq b' := \min_{\ell \neq s} |\lambda_\ell - \lambda_s| > 0$ by (2.6), and we argue similarly to the first case to obtain (4.4), with $C_2 = (1+\delta) \max\{\frac{2}{b'}, \frac{4C_1}{b'}\}$. \square

Proof of Proposition 3.1. Let

$$P_\omega := \{(\mathbf{i}, \mathbf{j}) \in P : \mathbf{i} \wedge \mathbf{j} = \omega\},$$

where $\omega = (\omega_1, \dots, \omega_k) \in \{1, \dots, m\}^k$ for some k . Denote

$$A_\omega(r) := \iint_{P_\omega} g_r(\mathbf{i}, \mathbf{j}) d\mu_2(\mathbf{i}, \mathbf{j}),$$

so that

$$A(r) = \sum_{k=N}^{\infty} \sum_{|\omega|=k} A_\omega(r). \quad (4.7)$$

We can apply Lemma 4.1 with $N = N(\varepsilon)$. Then $F_\varepsilon \subset F$ and $G_\varepsilon \subset G$ by (3.8) and (3.9), so for $\mathbf{i}, \mathbf{j} \in G_\varepsilon$, with $|\mathbf{i} \wedge \mathbf{j}| = k \geq N$, we have

$$g_r(\mathbf{i}, \mathbf{j}) = \eta_\infty \{ \mathbf{y} \in F_\varepsilon : \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) < r \} \leq C_2(1 + \delta)^k \rho^{-k} r.$$

Thus for $|\omega| = k \geq N$,

$$A_\omega(r) \leq C_2(1 + \delta)^k \rho^{-k} r \cdot (\mu \times \mu)\{(\mathbf{i}, \mathbf{j}) : \mathbf{i} \wedge \mathbf{j} = \omega\}. \quad (4.8)$$

On the other hand,

$$(\mu \times \mu)\{(\mathbf{i}, \mathbf{j}) : \mathbf{i} \wedge \mathbf{j} = \omega\} \leq \mu([\omega])^2 = \mu[\mathbf{i}, k] \cdot \mu([\omega]) \leq \theta^k \mu([\omega]), \quad (4.9)$$

in view of (3.9). Combining this with (4.8) and (4.7) we obtain

$$A(r) \leq C_2 \sum_{k \geq N} \sum_{|\omega|=k} (1 + \delta)^k \theta^k \rho^{-k} \mu([\omega]) \cdot r < \text{const} \cdot r,$$

where we used that $\sum_{|\omega|=k} \mu([\omega]) = 1$ and (3.7). The proof is complete. \square

5. PROOF OF THEOREM 2.1(B)

Fix $\varepsilon \in (0, 1)$; our goal is to prove (3.5) assuming that $-h(\mu) \geq \chi(\mu)$.

ESTIMATE FROM BELOW. Fix an arbitrary $\alpha < h(\mu)/|\chi(\mu)|$; it is enough to prove that

$$\dim_H(\nu_{\mathbf{y}}^\varepsilon) \geq \alpha \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y} \in F_\varepsilon. \quad (5.1)$$

We can find $\theta, \rho, \delta > 0$ such that

$$\alpha < \frac{\log \theta}{\log((1 + \delta)^{-1} \rho)}, \quad |\chi(\mu)| < -\log \rho, \quad \text{and} \quad h(\mu) > -\log \theta. \quad (5.2)$$

Similarly to the proof of Theorem 2.1(a), we can find $N = N(\varepsilon, \delta, \theta, \rho)$ such that

$$(1 + \delta)^{-n} \leq y_1 \cdots y_n \leq (1 + \delta)^n \quad \text{for all } n \geq N, \mathbf{y} \in F_\varepsilon$$

and

$$\mu[\mathbf{i}, n] < \theta^n \quad \text{and} \quad \rho^n < \lambda_{i_1 \dots i_n}, \quad \text{for all } n \geq N, \mathbf{i} \in G_\varepsilon.$$

We will use the decomposition (3.10) again.

By Frostman's Theorem, see [4, Theorem 4.13], for any Borel measure ν on the line,

$$\dim_H(\nu) \geq \sup \left\{ \alpha > 0 : \iint_{\mathbb{R}^2} \frac{d\nu(\xi) d\nu(\zeta)}{|\xi - \zeta|^\alpha} < \infty \right\}. \quad (5.3)$$

Thus the desired estimate (5.1) will follow by Fubini's Theorem, if we show that

$$\mathcal{S} := \sum_{|\omega|=N} \int_{F_\varepsilon} \iint_{\mathbb{R}^2} |\xi - \zeta|^{-\alpha} d\nu_{\mathbf{y},\omega}^\varepsilon(\xi) d\nu_{\mathbf{y},\omega}^\varepsilon(\zeta) d\eta_\infty(\mathbf{y}) < \infty. \quad (5.4)$$

After changing the variables and reversing the order of integration we obtain

$$\mathcal{S} = \sum_{k=N}^{\infty} \sum_{|\omega|=k} \iint_{P_\omega} \int_{F_\varepsilon} \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j})^{-\alpha} d\eta_\infty(\mathbf{y}) d\mu_2(\mathbf{i}, \mathbf{j}), \quad (5.5)$$

where again

$$P_\omega = \{(\mathbf{i}, \mathbf{j}) \in G_\varepsilon \times G_\varepsilon : \mathbf{i} \wedge \mathbf{j} = \omega\}.$$

Suppose that $|\mathbf{i} \wedge \mathbf{j}| = k$. The inner integral in (5.5) is equal to

$$\alpha \int_0^\infty \eta_\infty\{\mathbf{y} \in F_\varepsilon : \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j}) \leq r\} r^{-\alpha-1} dr = \int_0^{(1+\delta)^{-k}\rho^k} + \int_{(1+\delta)^{-k}\rho^k}^\infty.$$

The first integral in the right-hand side is estimated by Lemma 4.1, and the second integral is estimated by the trivial estimate $\eta\{\cdot\} \leq 1$ yielding the inequality

$$\int_{F_\varepsilon} \varphi(\mathbf{y}, \mathbf{i}, \mathbf{j})^{-\alpha} d\eta_\infty(\mathbf{y}) \leq \text{const} \cdot [(1+\delta)\rho^{-1}]^{\alpha k}.$$

Substituting this into (5.5) we obtain

$$\mathcal{S} \leq \text{const} \cdot \sum_{k=N}^{\infty} \sum_{|\omega|=k} [(1+\delta)\rho^{-1}]^{\alpha k} \mu_2\{(\mathbf{i}, \mathbf{j}) : \mathbf{i} \wedge \mathbf{j} = \omega\}.$$

Now we can apply (4.9) to get

$$\mathcal{S} \leq \text{const} \cdot \sum_{k=N} [(1+\delta)\rho^{-1}]^{\alpha k} \theta^k < \infty,$$

by (5.2).

ESTIMATE FROM ABOVE. Dimension estimates from above are fairly standard. This is also the case here, although there are technical complications because of the generality of our set-up. Note that we obtain the upper bound for all, rather than almost all, $\mathbf{y} \in F_\varepsilon$, and the distribution of y_i 's is irrelevant here. (Recall that F_ε was defined at the beginning of Section 3.) A similar upper bound for (possibly nonlinear, but non-random) contracting on average IFS was obtained in [14, 6].

Fix an arbitrary $\alpha > h(\mu)/|\chi(\mu)|$; it is enough to prove that

$$\forall \varepsilon > 0, \dim_H(\nu_{\mathbf{y}}) \leq \alpha \text{ for all } \mathbf{y} \in F_\varepsilon. \quad (5.6)$$

We fix $\varepsilon > 0$ and $\mathbf{y} \in F_\varepsilon$ for the rest of this proof.

Now let $\gamma > 0$ and consider G_γ , with $\mu(G_\gamma) > 1 - \gamma$, such that the convergence in (3.2) and (3.3) is uniform on G_γ . Further, let $C_3 > 0$ be such that

$$\mu(\Omega_\gamma) \geq 1 - \gamma, \quad \text{where } \Omega_\gamma := \{\mathbf{i} \in \Sigma : |\Pi_{\mathbf{y}}(\mathbf{i})| \leq C_3\}. \quad (5.7)$$

Consider

$$A_\gamma^n := \{\mathbf{i} \in G_\gamma \cap \sigma^{-n}\Omega_\gamma : \mu([\mathbf{i}, n] \cap \sigma^{-n}\Omega_\gamma) \geq 0.5 \cdot \mu([\mathbf{i}, n])\}. \quad (5.8)$$

We claim that $\mu(A_\gamma^n) \geq 1 - 4\gamma$ for all $n \in \mathbb{N}$. Indeed, $\mu(G_\gamma \cap \sigma^{-n}\Omega_\gamma) \geq 1 - 2\gamma$, and the measure of the complement of the set of \mathbf{i} satisfying the inequality in (5.8) equals

$$\mu\{\mathbf{i} \in \Sigma : \mu([\mathbf{i}, n] \cap (\sigma^{-n}\Omega_\gamma)^c) \geq 0.5 \cdot \mu([\mathbf{i}, n])\} \leq 2\mu((\sigma^{-n}\Omega_\gamma)^c) \leq 2\gamma,$$

where we used that μ is σ -invariant in the last step. It follows that

$$\mu(H_\gamma) \geq 1 - 4\gamma, \quad \text{where } H_\gamma := \limsup(A_\gamma^n) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_\gamma^n. \quad (5.9)$$

Recall that our goal is to prove $\dim_H(\nu_{\mathbf{y}}) \leq \alpha$. Billingsley's Theorem (see [5, p.171]) states that

$$\dim_H(\nu_{\mathbf{y}}) = \nu_{\mathbf{y}}\text{-ess sup} \left\{ \liminf_{r \downarrow 0} \frac{\log \nu_{\mathbf{y}}[x - r, x + r]}{\log(2r)} \right\}.$$

Thus it is enough to verify that

$$\liminf_{r \downarrow 0} \frac{\log \nu_{\mathbf{y}}[x - r, x + r]}{\log(2r)} \leq \alpha \quad (5.10)$$

for $\nu_{\mathbf{y}}$ a.e. x . Since $\nu_{\mathbf{y}} = \mu \circ \Pi_{\mathbf{y}}^{-1}$ and in view of (5.9), this will follow if we prove (5.10) for all $x \in \Pi_{\mathbf{y}}(H_\gamma)$, for every $\gamma > 0$. To this end, let us fix $\gamma > 0$ and $x = \Pi_{\mathbf{y}}(\mathbf{i})$ for some $\mathbf{i} \in H_\gamma$. Since $\mathbf{i} \in H_\gamma$, there exists a sequence $n_k \rightarrow \infty$ such that

$$\mu([\mathbf{i}, n_k] \cap \sigma^{-n_k}\Omega_\gamma) \geq 0.5 \cdot \mu[\mathbf{i}, n_k], \quad \forall k \in \mathbb{N}. \quad (5.11)$$

Since $\alpha > h(\mu)/|\chi(\mu)|$, we can find $\theta, \rho, \delta > 0$ such that

$$\alpha > \frac{\log \theta}{\log((1 + \delta)\rho)}, \quad |\chi(\mu)| > -\log \rho, \quad \text{and } h(\mu) < -\log \theta. \quad (5.12)$$

Similarly to the proof of Theorem 2.1(a), we can find N such that (3.8) holds and

$$\mu[\mathbf{i}, n] > \theta^n \quad \text{and} \quad \rho^n > \lambda_{i_1 \dots i_n}, \quad \text{for all } n \geq N, \mathbf{i} \in G_\gamma. \quad (5.13)$$

Let $r_k = 2C_3\rho^{n_k}(1 + \delta)^{n_k}$. We claim that for all k sufficiently large,

$$\nu_{\mathbf{y}}[x - r_k, x + r_k] = \mu\{\mathbf{j} : |\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})| \leq r_k\} \geq 0.5 \cdot \mu[\mathbf{i}, n_k] \quad (5.14)$$

(the equality here is by definition; the claim is the inequality). Indeed, let $\mathbf{j} \in [\mathbf{i}, n_k] \cap \sigma^{-n_k}\Omega_\gamma$. Then for k sufficiently large (so that $n_k \geq N$), we have

$$|\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})| = \lambda_{i_1 \dots i_{n_k}} y_{1 \dots n} \cdot |\Pi_{\mathbf{y}}(\sigma^{n_k}\mathbf{i}) - \Pi_{\mathbf{y}}(\sigma^{n_k}\mathbf{j})| \leq 2C_3\rho^{n_k}(1 + \delta)^{n_k} = r_k,$$

using (5.13), (3.8) and the fact that $\sigma^{n_k}\mathbf{i}, \sigma^{n_k}\mathbf{j} \in \Omega_\gamma$, where Ω_γ is defined by (5.7). This, combined with (5.11), proves (5.14). Now, keeping in mind that the numerator and denominator below are negative, we obtain

$$\begin{aligned} \liminf_{k \rightarrow \infty} \frac{\log \nu[x - r_k, x + r_k]}{\log(2r_k)} &\leq \liminf_{k \rightarrow \infty} \frac{\log \mu[\mathbf{i}, n_k] - \log 2}{\log(2C_3) + n_k \log((1 + \delta)\rho)} \\ &\leq \lim_{k \rightarrow \infty} \frac{n_k \log \theta}{n_k \log((1 + \delta)\rho)} < \alpha, \end{aligned}$$

where we used (5.13) and (5.12). The proof is complete. \square

6. PROOFS OF OTHER RESULTS

6.1. Method of Kahane-Salem. Here we prove Theorem 2.3 using a variant of the approach from [7]. Recall that for a finite measure ν on \mathbb{R} its Fourier transform is defined by $\widehat{\nu}(\xi) = \int_{\mathbb{R}} e^{it\xi} d\nu(t)$.

Definition 6.1. For a finite measure ν on \mathbb{R} , its *Sobolev dimension* is defined as

$$\dim_s(\nu) = \sup \left\{ \alpha \in \mathbb{R} : \mathcal{E}_\alpha(\nu) = \int_{\mathbb{R}} |\widehat{\nu}(\xi)|^2 (1 + |\xi|)^{\alpha-1} d\xi < \infty \right\}. \quad (6.1)$$

Remark 6.2. If $\dim_s(\nu) < 1$, then $\dim_s(\nu)$ is also known as the correlation dimension of the measure ν . If $\mathcal{E}_\alpha(\nu) < \infty$ for $\alpha > 1$, then $\nu \ll \mathcal{L}$, and its density is said to have the fractional derivative of order $(\alpha - 1)/2$ in $L^2(\mathbb{R})$. If $\mathcal{E}_1(\nu) < \infty$, then ν has a density in $L^2(\mathbb{R})$ (this is just Plancherel's Theorem), and if $\dim_s(\nu) > 2$, then ν has a continuous density, see e.g. [1, Th. 1.2.4].

Theorem 6.3. Let $\nu_{\mathbf{y}}$ be the conditional distribution of the sum (2.3) given $\mathbf{y} = (y_1, y_2, \dots)$, defined by (2.4). We assume that (2.11) and (2.5) are satisfied, $\mu =$

$(p_1, \dots, p_m)^\mathbb{N}$ and $\lambda_i = \lambda$ for all $i \leq m$. Suppose that $\chi(\eta) = \log \lambda + \mathbb{E}[\log Y] < 0$ and denote $\beta = \sum_{j=1}^m p_j^2$. Then

$$\dim_s(\nu_{\mathbf{y}}) \geq |\log \beta| / |\chi(\eta)| \quad \text{for } \eta_\infty \text{ a.e. } \mathbf{y}.$$

In particular, if $0 > \chi(\eta) > \log \beta$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$ with a density in $L^2(\mathbb{R})$ for η_∞ a.e. \mathbf{y} . If $0 > \chi(\eta) > \frac{1}{2} \log \beta$, then $\nu_{\mathbf{y}} \ll \mathcal{L}$ with a continuous density for η_∞ a.e. \mathbf{y} .

In view of Remark 6.2, Theorem 2.3 is contained in Theorem 6.3.

Proof of Theorem 6.3. Since $\lambda_i = \lambda$ for all $i \leq m$, we can assume without loss of generality that $\lambda = 1$ (just replace Y with λY). Then $\chi(\eta) = \mathbb{E}[\log Y]$. Our goal is to prove that for every $\alpha < \frac{|\log \beta|}{|\chi(\eta)|}$,

$$\int_{\mathbb{R}} |\widehat{\nu}_{\mathbf{y}}(\xi)|^2 (1 + |\xi|)^{\alpha-1} d\xi < \infty, \quad (6.2)$$

for η_∞ a.e. \mathbf{y} . Fix $\alpha < \frac{|\log \beta|}{|\chi(\eta)|}$ for the rest of the proof. By the Law of Large Numbers and Egorov's Theorem, for any $\varepsilon > 0$ we can find $F_\varepsilon \subset \mathbb{R}^\mathbb{N}$ such that $\eta_\infty(F_\varepsilon) > 1 - \varepsilon$ and $(y_{1\dots n})^{1/n} \rightarrow e^{\chi(\eta)}$ uniformly on F_ε . It suffices to verify (6.2) for η_∞ a.e. $\mathbf{y} \in F_\varepsilon$, for an arbitrary $\varepsilon > 0$. Fix $\varepsilon > 0$ for the rest of the proof. The result will follow by Fubini's Theorem if we can show that

$$\int_{F_\varepsilon} \int_{\mathbb{R}} |\widehat{\nu}_{\mathbf{y}}(\xi)|^2 (1 + |\xi|)^{\alpha-1} d\xi d\eta_\infty(\mathbf{y}) < \infty.$$

Recall that $\nu_{\mathbf{y}}$ is the conditional distribution of X in (2.10) given \mathbf{y} , with $\lambda = 1$, which can be viewed as a sum of independent discrete random variables. Thus, $\nu_{\mathbf{y}}$ is the infinite convolution product

$$\nu_{\mathbf{y}} = * \prod_{n=1}^{\infty} \left(\sum_{j=1}^m p_j \delta_{d_j y_{1\dots(n-1)}} \right),$$

where δ is the Dirac's delta. Its Fourier transform is

$$\widehat{\nu}_{\mathbf{y}}(\xi) = \prod_{n=1}^{\infty} \sum_{j=1}^m p_j e^{i d_j y_{1\dots(n-1)} \xi} =: \prod_{n=1}^{\infty} \psi_n(\mathbf{y}, \xi). \quad (6.3)$$

Now the argument essentially follows the proof of [7, Théorème II]. We have

$$\begin{aligned} |\psi_n(\mathbf{y}, \xi)|^2 &= \sum_{j,k=1}^m p_j p_k e^{i(d_j - d_k) y_{1\dots(n-1)} \xi} \\ &= \sum_{j=1}^m p_j^2 + \sum_{j \neq k} p_j p_k e^{i(d_j - d_k) y_{1\dots(n-1)} \xi}. \end{aligned}$$

Clearly,

$$|\widehat{\nu}_{\mathbf{y}}(\xi)|^2 \leq \prod_{n=1}^{\ell+1} |\psi_n(\mathbf{y}, \xi)|^2 =: f_{\xi, \ell+1}.$$

Denote by F_ε^n the projection of F_ε onto \mathbb{R}^n (the first n coordinates). Then, since $f_{\xi, \ell+1}$ depends only on y_1, \dots, y_ℓ , we obtain

$$\begin{aligned} \int_{F_\varepsilon} f_{\xi, \ell+1} d\eta_\infty(\mathbf{y}) &= \int_{F_\varepsilon^\ell} f_{\xi, \ell+1} d\eta(y_1) \dots d\eta(y_\ell) \\ &\leq \int_{F_\varepsilon^{\ell-1}} f_{\xi, \ell} d\eta(y_1) \dots d\eta(y_{\ell-1}) \int_{\mathbb{R}} |\psi_{\ell+1}(\mathbf{y}, \xi)|^2 d\eta(y_\ell). \end{aligned} \quad (6.4)$$

Recall that $\beta = \sum_{j=1}^m p_j^2$, so

$$\begin{aligned} \int_{\mathbb{R}} |\psi_{\ell+1}(\mathbf{y}, \xi)|^2 d\eta(y_\ell) &= \beta + \sum_{j \neq k} p_j p_k \int_{\mathbb{R}} e^{i(d_j - d_k)y_{1\dots(n-1)}\xi} d\eta(y_\ell) \\ &= \beta + \sum_{j \neq k} p_j p_k \widehat{\eta}((d_j - d_k)y_{1\dots(n-1)}\xi). \end{aligned} \quad (6.5)$$

Integration by parts (see e.g. [8, p. 25]) shows that the Fourier transform of a compactly supported function of bounded variation is bounded above by $c|t|^{-1}$. Since $|d_j - d_k| \geq b > 0$ for $j \neq k$, we obtain that for some $C > 0$,

$$\int_{\mathbb{R}} |\psi_{\ell+1}(\mathbf{y}, \xi)|^2 d\eta(y_\ell) \leq \beta \left(1 + \frac{C}{y_{1\dots(\ell-1)}|\xi|} \right) \quad (6.6)$$

Recall that $\alpha < |\log \beta|/|\chi(\eta)|$; choose $\rho < e^{\chi(\eta)}$ such that $\alpha < |\log \beta|/|\chi(\eta)|$. Since $y_{1\dots n}^{1/n}$ converges to $e^{\chi(\eta)}$ uniformly on F_ε , we can find $N \in \mathbb{N}$ such that

$$y_{1\dots n} \geq (n+1)^2 \rho^{n+1} \quad \forall n \geq N, \quad \forall \mathbf{y} \in F_\varepsilon.$$

It follows from (6.4) and (6.6) that for $\ell \geq N+1$,

$$\begin{aligned} \int_{F_\varepsilon} f_{\xi, \ell+1} d\eta_\infty(\mathbf{y}) &\leq \int_{F_\varepsilon} f_{\xi, \ell} d\eta_\infty(\mathbf{y}) \cdot \beta \left(1 + \frac{C}{\ell^2 \rho^\ell |\xi|} \right) \\ &\leq \int_{F_\varepsilon} f_{\xi, \ell} d\eta_\infty(\mathbf{y}) \cdot \beta(1 + C\ell^{-2}), \end{aligned} \quad (6.7)$$

provided that $|\xi| \geq \rho^{-\ell}$. Clearly this condition holds for $\ell' < \ell$ if it holds for ℓ , so we can iterate (6.7) to obtain, assuming $|\xi| \geq \rho^{-\ell}$:

$$\int_{F_\varepsilon} f_{\xi, \ell+1} d\eta_\infty(\mathbf{y}) \leq \int_{F_\varepsilon} f_{\xi, N} d\eta_\infty(\mathbf{y}) \cdot \beta^{\ell+1-N} \prod_{k=N}^{\ell} (1 + Ck^{-2}) \leq C' \beta^\ell,$$

where $C' > 0$ depends on N but not on ξ . We conclude that

$$\int_{F_\varepsilon} |\widehat{\nu}_{\mathbf{y}}(\xi)|^2 d\eta_\infty(\mathbf{y}) \leq C'' |\xi|^{\log \beta / |\log \rho|}.$$

But $\log \beta / |\log \rho| < -\alpha$, hence

$$\int_{\mathbb{R}} \left(\int_{F_\varepsilon} |\widehat{\nu}_{\mathbf{y}}(\xi)|^2 d\eta_\infty(\mathbf{y}) \right) (1 + |\xi|)^{\alpha-1} d\xi < \infty,$$

and the proof is complete. \square

6.2. Proof of Proposition 2.2. Consider the probability space $\Sigma \times \mathbb{R}^{\mathbb{N}}$ with the measure $\mathbb{P} := \mu \times \eta_\infty$. Under the assumptions of the proposition, $Z_n := \log p_{i_n} - \log \lambda_{i_n} - \log y_n$ are i.i.d. non-constant random variables with mean zero. Let

$$B_n := \left\{ (\mathbf{i}, \mathbf{y}) : \sum_{j=1}^n Z_j > \sqrt{n} \right\} = \left\{ (\mathbf{i}, \mathbf{y}) : \frac{p_{i_1 \dots i_n}}{\lambda_{i_1 \dots i_n} y_{1 \dots n}} > e^{\sqrt{n}} \right\}.$$

By the Law of Iterated Logarithm, we have

$$\mathbb{P}(\limsup B_n) = 1. \tag{6.8}$$

For $(\mathbf{i}, \mathbf{y}) \in \limsup B_n$ and $k \in \mathbb{N}$ let

$$\tau_k = \tau_k(\mathbf{i}, \mathbf{y}) = \min\{n \geq k : (\mathbf{i}, \mathbf{y}) \in B_n\},$$

which is well-defined and finite. Let

$$A_n = \{(\mathbf{i}, \mathbf{y}) : |\Pi_{\sigma^n \mathbf{y}}(\sigma^n \mathbf{i})| \leq n\};$$

note that A_n is independent of $i_1, \dots, i_n, y_1, \dots, y_n$. Since the probability \mathbb{P} is σ -invariant and $\Pi_{\mathbf{y}}(\mathbf{i})$ is finite a.s., we have $\mathbb{P}(A_n) = \mathbb{P}\{(\mathbf{i}, \mathbf{y}) : \Pi_{\mathbf{y}}(\mathbf{i}) \leq n\} \uparrow 1$.

Now,

$$\begin{aligned} \mathbb{P} \left[\bigcup_{n \geq k} (A_n \cap B_n) \right] &\geq \mathbb{P} \left(\bigcup_{n \geq k} [A_n \cap \{\tau_k = n\}] \right) \\ &= \sum_{n=k}^{\infty} \mathbb{P}\{\tau_k = n\} \cdot \mathbb{P}(A_n) \geq \mathbb{P}(A_k) \rightarrow 1, \end{aligned}$$

as $k \rightarrow \infty$. In the last displayed line we used that A_n and $\{\tau_k = n\}$ are independent events and that $\sum_{n=k}^{\infty} \mathbb{P}\{\tau_k = n\} = 1$ by (6.8). It follows that

$$\mathbb{P}(\limsup(A_n \cap B_n)) = 1.$$

By Fubini, there exists $\Omega \subset \mathbb{R}^{\mathbb{N}}$ such that $\eta_{\infty}(\Omega) = 1$ and

$$\Sigma_{\mathbf{y}} := \{\mathbf{i} \in \Sigma : (\mathbf{i}, \mathbf{y}) \in \limsup(A_n \cap B_n)\}$$

has $\mu(\Sigma_{\mathbf{y}}) = 1$ for every $\mathbf{y} \in \Omega$. We claim that $\mathcal{L}(\Pi_{\mathbf{y}}(\Sigma_{\mathbf{y}})) = 0$ for every $\mathbf{y} \in \Omega$, which will imply that $\nu_{\mathbf{y}} = \mu \circ \Pi_{\mathbf{y}}^{-1} \perp \mathcal{L}$. Fix $\mathbf{y} \in \Omega$ and $k \in \mathbb{N}$. Observe that $\Sigma_{\mathbf{y}} \subset \bigcup_{n \geq k} \{\mathbf{i} : (\mathbf{i}, \mathbf{y}) \in A_n \cap B_n\}$. For any $(\mathbf{i}, \mathbf{y}) \in A_n \cap B_n$ and $(\mathbf{j}, \mathbf{y}) \in [\mathbf{i}, n] \cap A_n$ we have

$$\begin{aligned} |\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})| &= \lambda_{i_1 \dots i_n} y_{1 \dots n} |\Pi_{\sigma^n \mathbf{y}}(\sigma^n \mathbf{i}) - \Pi_{\sigma^n \mathbf{y}}(\sigma^n \mathbf{j})| \\ &\leq 2n \lambda_{i_1 \dots i_n} y_{1 \dots n} \\ &\leq 2n e^{-\sqrt{n}} p_{i_1 \dots i_n}. \end{aligned}$$

Here we used first that $(\mathbf{i}, \mathbf{y}), (\mathbf{j}, \mathbf{y}) \in A_n$, and then that $(\mathbf{i}, \mathbf{y}) \in B_n$. Summing over all cylinders of length n (using that $\sum_{i_1 \dots i_n} p_{i_1 \dots i_n} = 1$), and then summing over n we obtain

$$\mathcal{L}(\Pi_{\mathbf{y}}(\Sigma_{\mathbf{y}})) \leq \sum_{n \geq k} 2n e^{-\sqrt{n}} \rightarrow 0, \quad \text{as } k \rightarrow \infty.$$

□

6.3. Proof of Corollaries 2.4-2.6. By the Variational Principle (see e.g. [17]), $h_{\text{top}}(\Gamma) = \sup_{\mu} h(\mu)$, where the supremum is over ergodic σ -invariant measures supported on Γ . Thus, Theorem 2.1(a) implies Corollary 2.4(a), and Theorem 2.1(b) implies the lower estimate for $\dim_H(S_{\Gamma}(\mathbf{y}))$ in Corollary 2.4(b).

In Corollary 2.5, we have $\Gamma = \Sigma$, for which the measure of maximal entropy is $(\frac{1}{m}, \dots, \frac{1}{m})^{\mathbb{N}}$. Part (a) of Corollary 2.5 then follows from Theorem 2.3(b) by the Variational Principle, and Corollary 2.5(b) is a special case of Corollary 2.4(b).

It remains to verify the upper estimate for $\dim_H(S_{\mathbf{y}})$ in Corollary 2.4(b). By the Law of Large Numbers and Egorov's Theorem, we can find $F_{\varepsilon} \subset \mathbb{R}^{\mathbb{N}}$ such that $\eta_{\infty}(F_{\varepsilon}) > 1 - \varepsilon$ and $y_{1 \dots n}^{1/n} \rightarrow e^{\chi(\eta)}$ uniformly for $\mathbf{y} \in F_{\varepsilon}$. Fix an arbitrary $\alpha > h_{\text{top}}(\Gamma)/|\chi(\eta)|$. It suffices to show that for every $\varepsilon > 0$ we have $\overline{\dim}_{\text{box}}(S_{\mathbf{y}}) < \alpha$, for a.e. $\mathbf{y} \in F_{\varepsilon}$. Here $\overline{\dim}_{\text{box}}$ denotes the upper box-counting dimension. Fix $\varepsilon > 0$ for the rest of the proof.

Let $\delta > 0$ be such that $\chi(\eta) + \delta < 0$ and $\alpha > h_{\text{top}}(\Gamma)/(|\chi(\eta)| - \delta)$. We can find $N \in \mathbb{N}$ such that

$$y_{1 \dots n} \leq \exp(n(\chi(\eta) + \delta)), \quad \forall n \geq N, \quad \forall \mathbf{y} \in F_{\varepsilon}.$$

Then for $\mathbf{i}, \mathbf{j} \in \Sigma$ such that $|\mathbf{i} \wedge \mathbf{j}| \geq n \geq N$, we have

$$\begin{aligned} |\Pi_{\mathbf{y}}(\mathbf{i}) - \Pi_{\mathbf{y}}(\mathbf{j})| &= \left| \sum_{\ell=n}^{\infty} y_{1\dots\ell} (d_{i_{\ell+1}} - d_{j_{\ell+1}}) \right| \\ &\leq 2d_{\max} \sum_{\ell=n}^{\infty} e^{\ell(\chi(\eta)+\delta)} = \frac{2d_{\max} e^{n(\chi(\eta)+\delta)}}{1 - e^{\chi(\eta)+\delta}}, \end{aligned}$$

where $d_{\max} := \max_{i \leq m} |d_i|$. It follows that the diameter of the set $\Pi_{\mathbf{y}}([\omega])$ for ω of length $n \geq N$ and $\mathbf{y} \in F_{\varepsilon}$ is bounded above by $\text{const} \cdot e^{n(\chi(\eta)+\delta)}$. Denote by $\#\mathcal{W}_n(\Gamma)$ the number of cylinders $[\omega]$ of length n such that $[\omega] \cap \Gamma \neq \emptyset$. We obtain a cover of $S_{\mathbf{y}}$ by $\#\mathcal{W}_n(\Gamma)$ intervals of length $\text{const} \cdot e^{n(\chi(\eta)+\delta)}$, hence

$$\overline{\dim}_{\text{box}}(S_{\mathbf{y}}) \leq \limsup_{n \rightarrow \infty} \frac{\log(\#\mathcal{W}_n(\Gamma))}{n(-\chi(\eta) - \delta)} = \frac{h_{\text{top}}(\Gamma)}{|\chi(\eta)| - \delta} < \alpha.$$

Here we used the definition of the upper box-counting dimension and the definition of topological entropy. The proof is complete. \square

Proof of Corollary 2.6. This essentially follows the proof of Proposition 2.2. Let $\mu = (\frac{1}{m}, \dots, \frac{1}{m})$. Then $Z_n = -\log m - \log \lambda - \log y_n$ are i.i.d. non-constant random variables with mean zero on $\mathbb{R}^{\mathbb{N}}$. Define $B'_n = \{\mathbf{y} : \sum_{j=1}^n Z_j > \sqrt{n}\}$. Then $\eta_{\infty}(\limsup B'_n) = 1$ by the Law of Iterated Logarithm. We can define $\tau_k = \tau_k(\mathbf{y})$ similarly to the proof of Proposition 2.2. Then let $A'_n = \{\mathbf{y} : |\Pi_{\sigma^n \mathbf{y}}(\sigma^n \mathbf{i})| \leq n \forall \mathbf{i} \in \Sigma\}$. We have $\eta_{\infty}(A'_n) \uparrow 1$ and $\eta_{\infty}(\limsup(A'_n \cap B'_n)) = 1$ repeating the argument in the proof of Proposition 2.2. Now we can take $\Sigma_{\mathbf{y}} = \Sigma$ and conclude as in the proof of Proposition 2.2, obtaining that $\mathcal{L}(S_{\mathbf{y}}) = \mathcal{L}(\Pi_{\mathbf{y}}(\Sigma)) = 0$ for all $\mathbf{y} \in \limsup(A'_n \cap B'_n)$. \square

7. OPEN QUESTIONS

Question 1. Is the condition $\log m > 2|\chi(\eta)|$ in Corollary 2.5(a) for the random set $S_{\mathbf{y}}$ to contain an interval almost surely, sharp? Perhaps, $\log m > |\chi(\eta)|$ is already sufficient? This would mean that as soon the random set $S_{\mathbf{y}}$ has positive Lebesgue measure, it has non-empty interior (almost surely). This is interesting, in particular, for the example considered by Arratia, see Proposition 1.2.

Question 2. The results on continuous density and intervals in random sets (see Theorem 2.3 and Corollary 2.5(a)) are obtained only in the case when μ is a product measure. Extend this to the general case of ergodic μ .

Question 3. In Theorem 6.3 we prove a lower bound for the a.s. value of the Sobolev dimension $\dim_s(\nu_{\mathbf{Y}})$. Is this actually an equality? If $|\log \beta| < |\chi(\eta)|$, then the matching upper bound can be obtained from the fact that the Sobolev dimension equals the correlation dimension when it is less than one, but what about the case $|\log \beta| \geq |\chi(\eta)|$?

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YUVAL PERES, DEPARTMENT OF STATISTICS, UNIVERSITY OF CALIFORNIA, BERKELEY
E-mail address: `peres@stat.Berkeley.edu`

KÁROLY SIMON, INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY OF BUDAPEST, H-1529 B.O.BOX 91, HUNGARY
E-mail address: `simonk@math.bme.hu`

BORIS SOLOMYAK, BOX 354350, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE WA 98195
E-mail address: `solomyak@math.washington.edu`