“MANDELBROT SET” FOR A PAIR OF LINEAR MAPS: THE LOCAL GEOMETRY

BORIS SOLOMYAK

ABSTRACT. We consider the iterated function system $\{\lambda z - 1, \lambda z + 1\}$ in the complex plane, for λ in the open unit disk. Let $\mathcal{M}$ be the set of λ such that the attractor of the IFS is connected. We discuss some topological and geometric properties of the set $\mathcal{M}$ and prove a new result about possible “corners” on its boundary. Some open problems and directions for further research are discussed as well.

1. Introduction and statement of results

Consider the family of iterated function systems (IFS) in the complex plane $\{\lambda z - 1, \lambda z + 1\}$ depending on a parameter $\lambda \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let A_λ denote the attractor of the IFS, that is, A_λ is the unique nonempty compact set such that

$$A_\lambda = (-1 + \lambda A_\lambda) \cup (1 + \lambda A_\lambda). \tag{1.1}$$

The set

$$\mathcal{M} := \{\lambda \in \mathbb{D} : A_\lambda \text{ is connected}\}$$

was studied by Barnsley and Harrington [4], who called it the “Mandelbrot set for the pair of linear maps,” by analogy with the classical Mandelbrot set in complex dynamics. It is discussed at length in Barnsley’s book [3, Ch.8], and studied in [8, 5, 7, 6, 10, 1]. It is shown in [4] that

$$\{\lambda \in \mathbb{D} : |\lambda| \geq 2^{-1/2}\} \subset \mathcal{M} \subset \{\lambda \in \mathbb{D} : |\lambda| \geq 1/2\}. \tag{1.2}$$

Bousch [5], using the ideas of [9], proved that $\mathcal{M}$ is connected and locally connected.

Supported in part by NSF grant #DMS 0099814

2000 *Mathematics Subject Classification.* Primary 28A80; Secondary 28A78, 37F45, 26C10.

The following description of $\mathcal{M}$ is well-known; it follows from the fact that A_λ is connected if and only if the sets in the right-hand side of (1.1) intersect:

$$\mathcal{M} = \left\{ \lambda \in \mathbb{D} : \ \exists \{a_k\}_1^\infty, \ a_k \in \{-1, 0, 1\}, \ 1 + \sum_{k=1}^{\infty} a_k \lambda^k = 0 \right\}. \tag{1.3}$$

From (1.3) it is easy to deduce that $\mathcal{M}$ is relatively closed in the unit disc $\mathbb{D}$.

The set $\mathcal{M}$ is clearly symmetric with respect to both axes, so we can always confine ourselves to the first quarter of the plane. See Figure 1, made by Christoph Bandt, which shows the part of $\mathcal{M}$ in $\{z : \ \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) \geq 0, |z| \leq 1/\sqrt{2}\}$.

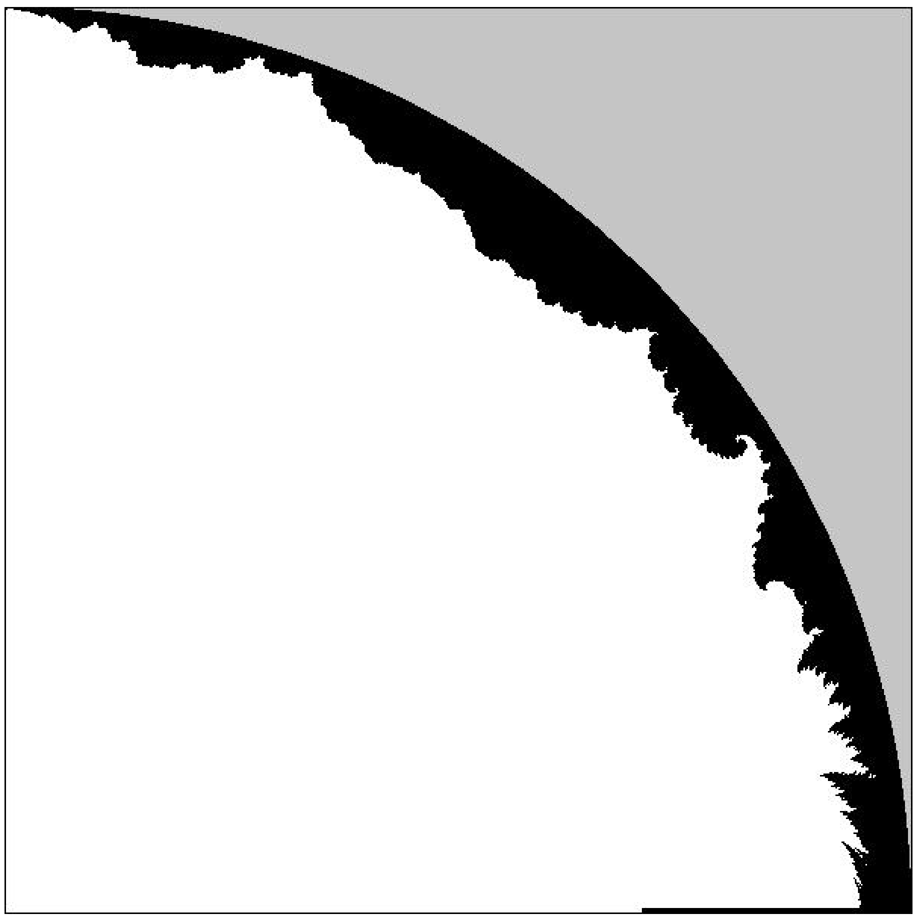

FIGURE 1. The "Mandelbrot set" $\mathcal{M}$

A peculiar feature of the set $\mathcal{M}$ is the "spike," or "antenna," on the positive real axis, from 0.5 to about .67, see [4, 10, 1] (of course, there is a symmetric one, on the negative real axis). It is exaggerated in the figure to make it visible. Bandt [1] conjectured that every *nonreal* point of the set $\mathcal{M}$ is a limit point of the interior points of $\mathcal{M}$, in other words, $\mathcal{M} \setminus \mathbb{R} \subset \text{clos}(\text{int}(\mathcal{M}))$. In joint work with Hui Xu, we obtained a partial result in this direction.

Theorem 1.1 ([11]). *The set $\mathcal{M} \cap \{\lambda \in \mathbb{D} : \; |\lambda| \le 2^{-1/2}\}$ has nonempty interior. Moreover, let $H := \{\lambda = \xi + i\eta : \; \frac{1}{3} \le |\lambda|^2 \le \frac{1}{2} \text{ and } 0 \le \xi \le \frac{3|\lambda|^2-1}{2}\}$; then*

$$\mathcal{M} \cap \text{int}(H) \subset \text{clos}(\text{int}(\mathcal{M})).$$

In this paper we continue the investigation of the set $\mathcal{M}$, focusing on the local geometry of the boundary $\partial\mathcal{M}$. We address the following question: does $\mathcal{M} \setminus \mathbb{R}$ have any "corners" ? By a "corner" we mean a point $z_0 \in \partial\mathcal{M} \setminus \mathbb{R}$ such that for some $r > 0$ and $\alpha < \pi$, the intersection $B(r, z_0) \cap \mathcal{M}$ is contained in a sector with the vertex at z_0 and the interior angle α (here $B(r, z_0)$ denotes the open disk of radius r centered at z_0). Figure 1 may suggest that "corners" exist, but we conjecture that there are none. So far, we have been able to prove a weaker result, showing that if "corners" do exist, they are very special.

Theorem 1.2. *Suppose that $z_0 \in \partial\mathcal{M} \setminus \mathbb{R}$ is such that $B(r, z_0) \cap \mathcal{M}$ is contained in a sector with the vertex at z_0 and the interior angle $\alpha < \pi$. Then*

(i) *z_0 is algebraic, and it has at most one algebraic conjugate in $B(2 \cdot 5^{-5/8}, 0)$ other than $\overline{z_0}$.*

(ii) *The angle $\arg(z_0)$ is rational modulo π. If $\arg(z_0) = \frac{\pi p}{q}$ for some $p, q \in \mathbb{N}$ mutually prime, then $q \ge 3$, and $\alpha \ge \frac{\pi(q-1)}{q}$.*

Remark. If there is a "corner," then its interior angle is at least $\frac{2\pi}{3}$, so there are no "antennas" other than those on the real axis. It also follows that for any $\varepsilon > 0$, there are finitely many "corners" with interior angle less than $\pi - \varepsilon$.

2. Proof of Theorem 1.2.

Let

$$\mathcal{B} = \left\{1 + \sum_{k=1}^{\infty} a_k z^k : \; a_k \in \{-1, 0, 1\} \, \forall k \right\}.$$

Note that $\mathcal{M} = \{\lambda \in \mathbb{D} : \; \exists f \in \mathcal{B}, \; f(\lambda) = 0\}$ by (1.3). Fix $z_0 \in \partial\mathcal{M} \setminus \mathbb{R}$. We can assume that $|z_0| < 1$; then $|z_0| \le 1/\sqrt{2}$ by (1.2). Fix $f \in \mathcal{B}$ such that $f(z_0) = 0$; let

$$f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k.$$

It follows from [2, Theorem 2] (taking $k = 4$) that functions of the class $\mathcal{B}$ cannot have a nonreal double zero of modulus less that $2 \cdot 5^{-5/8} \approx .73143$. Thus,

$$f'(z_0) \neq 0. \tag{2.1}$$

For any $n \in \mathbb{N}$ we have

$$\begin{aligned} a_n = 0 \quad &\Rightarrow \quad f(z) \pm z^n \in \mathcal{B}; \\ a_n = 1 \quad &\Rightarrow \quad f(z) - z^n \in \mathcal{B}; \\ a_n = -1 \quad &\Rightarrow \quad f(z) + z^n \in \mathcal{B}. \end{aligned} \tag{2.2}$$

A simple application of Rouché's Theorem shows that $g(z) = f(z) + bz^n$, for $b \in \{-1, 1\}$, has a zero z_1 close to z_0, and $z_1 \in \mathcal{M}$ for the appropriate choice of the sign. We need to know the approximate location of this zero. A good guess is provided by Newton's method, which yields, for n sufficiently large, in view of (2.1),

$$z_1 = z_0 - \frac{bz_0^n}{g'(z_0)} \approx z_0 - \frac{bz_0^n}{f'(z_0)}\,.$$

The needed estimates are contained in the following lemma. We introduce notation for a sector:

$$S(z_0, \theta, \alpha) := \{z = z_0 + re^{i\phi},\ r \geq 0,\ \phi \in [\theta - \alpha/2, \theta + \alpha/2]\}.$$

Lemma 2.1. *For any $\varepsilon > 0$, there exist $C_1 > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ and all $b \in \{-1, 1\}$, the function $f(z) + bz^n$ has a zero in*

$$B(C_1|z_0|^n, z_0) \cap S(z_0, -b \arg(z_0^n/f'(z_0)), \varepsilon).$$

Next we deduce the theorem, postponing the proof of the lemma to the next section.

Assume that there exist $r > 0$, $\alpha \in (0, \pi)$, and $\theta \in [0, 2\pi)$, such that

$$B(r, z_0) \cap \mathcal{M} \subset S(z_0, \theta, \alpha). \tag{2.3}$$

As explained above, for any $n \in \mathbb{N}$ there exists $b \in \{-1, 1\}$ such that the zeros of $f(z) + bz^n$ lie in $\mathcal{M}$. By Lemma 2.1 and (2.3), it follows that for any $\varepsilon > 0$, for all n sufficiently large, we have

$$S(z_0, \theta, \alpha) \cap \Big(S(z_0, \arg(z_0^n/f'(z_0)), \varepsilon) \cup S(z_0, -\arg(z_0^n/f'(z_0)), \varepsilon)\Big) \neq \{z_0\}. \tag{2.4}$$

This immediately implies that $\arg(z_0)/\pi \in \mathbb{Q}$ since otherwise $\{\arg(z_0^n/f'(z_0)) : n \in \mathbb{N}\}$ is dense in $[0, 2\pi)$, contradicting (2.4), see Figure 2.

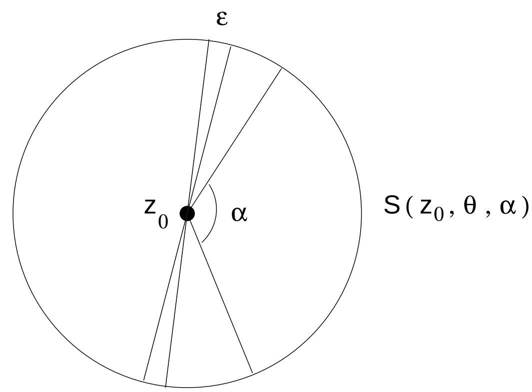

FIGURE 2

Let $\arg(z_0) = \frac{\pi p}{q}$ for some $0 < p < q$, with $p, q \in \mathbb{N}$ mutually prime (we can of course assume that $\mathrm{Im}(z_0) > 0$). Observe that $q \neq 2$. Indeed, otherwise we would have $z_0 = i|z_0|$. It is easy to verify that

$$\{iy : \ y > 0\} \cap \mathcal{M} = [2^{-1/2}i, i),$$

and $2^{-1/2}i$ is not a "corner," in view of (1.2).

Let $0 < \varepsilon < \frac{\pi - \alpha}{2}$. Then for any $n \in \mathbb{N}$ and any $\phi \in [0, 2\pi)$,

$$\text{either} \ \ S(z_0, \phi, \varepsilon) \cap S(z_0, \theta, \alpha) = \{z_0\}, \ \ \text{or} \ \ S(z_0, -\phi, \varepsilon) \cap S(z_0, \theta, \alpha) = \{z_0\},$$

In view of Lemma 2.1 and (2.2), for all n sufficiently large, $a_n \in \{-1, 1\}$ is uniquely determined by the condition

$$S(z_0, +a_n \arg(z_0^n / f'(z_0)), \varepsilon) \cap S(z_0, \theta, \alpha) \neq \{z_0\}.$$

The sequence $\{\arg(z_0^n/f'(z_0))\}_{n\geq 1}$ is periodic, so $\{a_n\}$ is eventually periodic. This implies that $f(z)$ is rational, hence z_0 is algebraic. (More precisely, there is a polynomial, with integer coefficients not greater than two in modulus and constant coefficient one, which has z_0 as a root.) All algebraic conjugates of z_0 are zeros of f, and every nonreal zero comes with a complex conjugate. By [2, Theorem 2], the number of zeros of f, that are less than $2 \cdot 5^{-5/8}$ in modulus, is less than four, which implies the claim about the conjugates in Theorem 1.2(i).

Finally, observe that the set $\{b \arg(z_0^n/f'(z_0)) : \ n \in \mathbb{N}, \ b \in \{-1, 1\}\}$ has cardinality $2q$ and intersects every sector with interior angle greater or equal to π/q. If $\alpha < \frac{\pi(q-1)}{q}$, then we can find n arbitrarily large and $b \in \{-1, 1\}$ such that

$$b \arg(z_0^n/f'(z_0)) \in (\theta + \pi/2 - \pi/2q, \ \ \theta + \pi/2 + \pi/2q)$$

which contradicts (2.4) for $\varepsilon > 0$ sufficiently small. The theorem is proved. $\square$

3. Proof of Lemma 2.1

First we need the following

Sublemma 3.1. *Suppose that $g(z)$ is analytic in $B(R, z_0)$, $|g(z_0)| \le \eta_1$, $|g'(z_0)| \ge \eta_2$, and $|g''(z)| \le L$ for all $z \in B(R, z_0)$. If $\eta_1 < \eta_2^2/2L$ and $2\eta_1/\eta_2 < R$, then there is a unique $z_1 \in B(2\eta_1/\eta_2, z_0)$ such that $g(z_1) = 0$.*

Proof. Since $|g''(z)| \le L$ for $z \in B(R, z_0)$, we have by Taylor's formula,

$$|g(z) - g(z_0) - g'(z_0)(z - z_0)| \le (L/2) \cdot |z - z_0|^2 \quad \text{for} \quad z \in B(R, z_0). \tag{3.1}$$

Notice that $\zeta = z_0 - \frac{g(z_0)}{g'(z_0)}$ is the unique zero of $p(z) := g(z_0) + g'(z_0)(z - z_0)$, and $|\zeta - z_0| = |\frac{g(z_0)}{g'(z_0)}| \le \frac{\eta_1}{\eta_2}$. On the circle $\{z : \ |z - z_0| = \frac{2\eta_1}{\eta_2}\}$ we have

$$|p(z)| \ge |g'(z_0)| \cdot |z - z_0| - |g(z_0)| \ge \eta_2 \cdot \frac{2\eta_1}{\eta_2} - \eta_1 = \eta_1.$$

Since $2\eta_1/\eta_2 < R$, the inequality (3.1), together with $\eta_1 < \eta_2^2/2L$, implies

$$|g(z) - p(z)| \le (L/2) \cdot (2\eta_1/\eta_2)^2 < \eta_1, \quad \text{for} \quad |z - z_0| = 2\eta_1/\eta_2,$$

so by Rouché's Theorem, $g(z)$ has a unique zero in $B(2\eta_1/\eta_2, z_0)$, as desired. $\square$

We are going to use the sublemma for $g(z) = f(z) + bz^n$, with n sufficiently large. Let $C_1 := 4/|f'(z_0)|$. Since $|z_0| \le 2^{-1/2}$, the disk $B(C_1|z_0|^n, z_0)$ is contained in $B(0.8,\ 0)$ for n sufficiently large; in this ball g is analytic and satisfies

$$|g''(z)| \le 4(1 - |z|)^{-3} \le L$$

for $L = 4 \cdot 5^3$. We have $|g(z_0)| = |z_0|^n$ and

$$|g'(z_0)| = |f'(z_0) + bnz_0^{n-1}| \ge |f'(z_0)|/2$$

for n sufficiently large (recall that $f'(z_0) \ne 0$). Let $R = 0.8 - 2^{-1/2}$, $\eta_1 = |z_0|^n$, and $\eta_2 = |f'(z_0)|/2$. Clearly, the assumptions of Sublemma 3.1 are satisfied for n sufficiently large, and we obtain that $g(z)$ has a zero $z_1 \in B(4|z_0|^n/|f'(z_0)|, z_0) = B(C_1|z_0|^n, z_0)$ (for n sufficiently large). Using (3.1) again, we get

$$|g(z_1) - g(z_0) - g'(z_0)(z_1 - z_0)| \le (L/2)|z_1 - z_0|^2,$$

and since $g(z_1) = 0$,

$$\begin{aligned} |z_1 - (z_0 - bz_0^n/g'(z_0))| &= |z_1 - (z_0 - g(z_0)/g'(z_0))| \\ &\le (2|g'(z_0)|)^{-1} L |z_1 - z_0|^2 \\ &\le |f'(z_0)|^{-1} L \cdot C_1^2 |z_0|^{2n}. \end{aligned} \tag{3.2}$$

Now observe that

$$\left|\frac{1}{g'(z_0)} - \frac{1}{f'(z_0)}\right| = \frac{n|z_0|^{n-1}}{|f'(z_0)g'(z_0)|} \le \frac{2n|z_0|^{n-1}}{|f'(z_0)|^2} \le C_2|z_0|^{(\gamma-1)n} \tag{3.3}$$

for some $C_2 = C_2(\gamma) > 0$ since $\gamma < 2$. Combining (3.3) with (3.2) yields that

$$z_1 \in B\Big(C_3|z_0|^{n\gamma},\ z_0 - \frac{bz_0^n}{f'(z_0)}\Big),$$

for some $C_3 > 0$. But $z_1 \in B(C_1|z_0|^n, z_0)$ as well, and since $\gamma > 1$, we see that for n sufficiently large, z_1 lies in the sector $S(z_0, -b \arg(z_0^n/f'(z_0)), \varepsilon)$, as desired. □

4. Concluding remarks and open questions

4.1. **"Corners" and spiral points.** We think that the set $\partial\mathcal{M} \setminus \mathbb{R}$ has no "corners." Theorem 1.2 implies that if there is a "corner" at $z_0 = \rho e^{i\phi}$, then ρ is algebraic and $e^{i\phi}$ is a root of unity. Perhaps one could rule them out by a a combination of algebraic and analytic tools.

Figure 1 suggests that many prominent points on the boundary of $\mathcal{M}$ are *spiral points.* For example, Bandt [1, Example 6.2] states that the solution $\lambda \approx 0.5739495 + 0.3689894i$ of the equation $1 - z + \frac{z^3}{1-z} = 0$ is the tip of the biggest spiral. It should be possible to check rigorously that λ is in $\partial\mathcal{M}$, and is in fact a spiral point.

4.2. **Local structure of $\mathcal{M}$ near the boundary.** It is mentioned in [3] that the local structure of $\mathcal{M}$ near $\lambda \in \partial\mathcal{M}$ resembles that of the corresponding attractor A_λ. While this is true to some extent, it appears that there is more resemblance with the set $A_\lambda\{-1, 0, 1\}$, attractor of the IFS consisting of three maps $\{\lambda z - 1, \lambda z, \lambda z + 1\}$. For one thing, if $\lambda \in \partial\mathcal{M} \setminus \mathbb{R}$, then it seems that $|\lambda| < 2^{-1/2}$, except for $\lambda = \pm 2^{-1/2} i$ (this is based on Figure 1 and other computer pictures; we certainly do not have a proof of this). Then, assuming $\mathrm{Re}(\lambda) \neq 0$, we have that A_λ has zero area and hence empty interior, whereas it is conjectured that λ is a limit point of interior points of $\mathcal{M}$ (and proved in Theorem 1.1 for λ sufficiently close to the imaginary axis). On the other hand, $A_\lambda\{-1, 0, 1\}$ has similarity dimension

$\log 3/\log|\lambda|$, which is greater than two for $|\lambda| \ge 1/\sqrt{3} \approx 0.577$. Bandt [1, Example 7.4] states that the point of minimum modulus on $\mathcal{M}\backslash\mathbb{R}$ has modulus about 0.6366, so $A_\lambda\{-1,0,1\}$ at least has a chance to have nonempty interior. Computer pictures of the sets $A_\lambda\{-1,0,1\}$ also support the idea that there is such a resemblance. There is a heuristic justification for the resemblance along the lines of [9], which runs as follows:

Let $\lambda \in \mathcal{M}$ and let $f(z) = 1 + \sum_{k=1}^\infty a_k z^k \in \mathcal{B}$ be such that $f(\lambda) = 0$. For any $n \in \mathbb{N}$ and any $\{b_j\}_{j=0}^\infty \in \{-1,0,1\}^\infty$, let $R_n(z) = \sum_{j=n}^\infty a_j z^j$. Consider

$$g(z) = f(z) - R_n(z) + z^n \sum_{j=0}^\infty b_j z^j \in \mathcal{B}.$$

For large n, the function $g(z)$ has a zero ζ close to λ, and similarly to the proof of Lemma 2.1, we can write

$$\zeta \approx \lambda - \frac{g(\lambda)}{g'(\lambda)} = \lambda - \frac{-R_n(\lambda) + \lambda^n \sum_{j=0}^\infty b_j \lambda^j}{g'(\lambda)} \approx \lambda - \frac{-R_n(\lambda) + \lambda^n \sum_{j=0}^\infty b_j \lambda^j}{f'(\lambda)}.$$

Since $\{\sum_{j=0}^\infty b_j \lambda^j :\ b_j \in \{-1,0,1\}\} = A_\lambda\{-1,0,1\}$, we see that $\mathcal{M}$ contains a subset which is "approximately" equal to

$$\lambda + \frac{R_n(\lambda)}{f'(\lambda)} - \frac{\lambda^n}{f'(\lambda)} A_\lambda\{-1,0,1\}.$$

Questions. Is it possible to make this correspondence rigorous? Which properties of $A_\lambda\{-1,0,1\}$ carry over into the local properties of $\mathcal{M}$? In particular, if $A_\lambda\{-1,0,1\}$ has nonempty interior, is λ necessarily a limit point of interior points of $\mathcal{M}$?

It would also be interesting to estimate the local dimension (Hausdorff or box-counting) of the boundary $\partial\mathcal{M}$, which seems to depend strongly on the argument of the point. Even if one can make a link with the sets $A_\lambda\{-1,0,1\}$, it does not necessarily help, since the latter is a self-similar set with large overlaps, and we do not know of any meaningful estimates of the dimension of its boundary.

Acknowledgment. The results of this paper were first reported at the ICM 2002 Satellite Conference on Fractal Geometry in Nanjing, China. I would like to thank the organizers, especially Professors Z. Y. Wen and W. Y. Su for their warm hospitality.

References

1. C. Bandt, On the Mandelbrot set for pairs of linear maps, *Nonlinearity* **15** (2002), 1127–1147.
2. F. Beaucoup, P. Borwein, D. W. Boyd, and C. Pinner, Multiple roots of $[-1, 1]$ power series, *J. London Math. Soc.(2)* **57** (1998), 135–147.
3. M. F. Barnsley, *Fractals Everywhere*, Academic Press, 1988.
4. M. F. Barnsley and A. N. Harrington, A Mandelbrot set for pairs of linear maps, *Phisica* **15D** (1985), 421–432.
5. T. Bousch, Connexité locale et par chemins hölderiens pour les systèmes itérés de fonctions, Preprint, 1993.
6. K.-H. Indlekofer, A. Járai, and I. Kátai, On some properties of attractors generated by iterated function systems, *Acta Sci. Math. (Szeged)* **60** (1995), 411–427.
7. K.-H. Indlekofer, I. Kátai, and P. Racsko, Some remarks on generalized number systems, *Acta Sci. Math. (Szeged)* **57** (1993), 543–553.
8. M. Morán, Chaotic control, *Iteration theory (Batschuns, 1992)*, 207–219, World Sci. Publishing, 1996.
9. A. M. Odlyzko and B. Poonen, Zeros of polynomials with 0,1 coefficients, *L'Enseignement Math.* **39** (1993), 317–348.
10. B. Solomyak, Measure and dimension of some fractal families, *Math. Proc. Cambridge Phil. Soc.* **124**, no. 3 (1998), 531–546.
11. B. Solomyak and H. Xu, On the "Mandelbrot set" for a pair of linear maps and complex Bernoulli convolutions, *Preprint,* 2002.

Department of Mathematics, University of Washington, Seattle, WA 98195-4350, USA. Email: solomyak@math.washington.edu