

# SPECTRA OF BERNOULLI CONVOLUTIONS AS MULTIPLIERS IN $L^p$ ON THE CIRCLE

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ABSTRACT. It is shown that the closure of the set of Fourier coefficients of the Bernoulli convolution  $\mu_\theta$  parameterized by a Pisot number  $\theta$  is countable. Combined with results of Salem and Sarnak, this proves that for every fixed  $\theta > 1$  the spectrum of the convolution operator  $f \mapsto \mu_\theta * f$  in  $L^p(S^1)$  (where  $S^1$  is the circle group) is countable and is the same for all  $p \in (1, \infty)$ , namely,  $\overline{\{\widehat{\mu_\theta}(n) : n \in \mathbb{Z}\}}$ . Our result answers the question raised by P. Sarnak in [8]. We also consider the sets  $\overline{\{\widehat{\mu_\theta}(rn) : n \in \mathbb{Z}\}}$  for  $r > 0$  which correspond to a linear change of variable for the measure. We show that such a set is still countable for all  $r \in \mathbb{Q}(\theta)$  but uncountable (a non-empty interval) for Lebesgue-a.e.  $r > 0$ .

## 1. INTRODUCTION AND MAIN RESULTS

Let  $\nu$  be a Borel probability measure on  $S^1 = \mathbb{R}/\mathbb{Z}$ , and let

$$T_\nu : L^p(S^1) \rightarrow L^p(S^1)$$

be the convolution operator, namely,  $T_\nu f := \nu * f$ . Put

$$\mathcal{F}_\nu := \{\widehat{\nu}(n) : n \in \mathbb{Z}\}.$$

It is shown by Sarnak [8] that if the closure  $\overline{\mathcal{F}_\nu}$  has capacity zero, then the following identity relation for the spectra of  $T_\nu$  in different  $L^p(S^1)$  is satisfied:

$$(1.1) \quad \text{sp}(T_\nu, L^p) \equiv \text{sp}(T_\nu, L^2) = \overline{\mathcal{F}_\nu}, \quad 1 < p < +\infty.$$

The present paper deals with the case when  $\nu$  is a Bernoulli convolution. Recall that for any  $\theta > 1$  the *Bernoulli convolution* parameterized

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by  $\theta$  is defined as follows:

$$\mu_\theta := * \prod_{k=0}^{\infty} \left( \frac{1}{2} \delta_{-\theta^{-k}} + \frac{1}{2} \delta_{\theta^{-k}} \right)$$

(where  $\delta_x$  denotes the Dirac  $\delta$ -measure at  $x$ ). This is a measure on the line; we use the same notation  $\mu_\theta$  for the induced measure on the circle. We have

$$(1.2) \quad \widehat{\mu}_\theta(t) = \prod_{k=0}^{\infty} \cos(2\pi\theta^{-k}t), \quad t \in \mathbb{R},$$

and the induced measure on the circle has Fourier coefficients  $\{\widehat{\mu}_\theta(n) : n \in \mathbb{Z}\}$ .

As is well known,  $\text{supp } \mu_\theta \subset I_\theta = [-\frac{\theta}{\theta-1}, \frac{\theta}{\theta-1}]$  for any  $\theta > 1$ . For  $\theta > 2$ , the measure  $\mu_\theta$  is usually called the *Cantor-Lebesgue measure* (parameterized by  $\theta$ ), and its support is a Cantor set with constant dissection ratio  $\theta$ . On the other hand,  $\text{supp } \mu_\theta = I_\theta$  for  $\theta \in (1, 2]$ .

In [7, Th II, p. 40] it is shown that unless  $\theta$  is a *Pisot number* (an algebraic integer greater than 1 whose conjugates are all less than 1 in modulus),  $\widehat{\mu}_\theta(t) \rightarrow 0$  as  $t \rightarrow +\infty$  along the reals, whence  $\overline{\mathcal{F}_{\mu_\theta}}$  is countable and thus, (1.1) is satisfied.

Sarnak [8] considered the case of the classical Cantor-Lebesgue measure ( $\theta = 3$ ), for which he proved that  $\overline{\mathcal{F}_{\mu_\theta}}$  is countable and therefore (1.1) holds. As stated in [8], the same approach can be applied to the case of an arbitrary integer  $\theta \geq 3$ , and the only case left is the irrational Pisot numbers  $\theta$ . The question about the limit points of the Fourier coefficients for this class of measures was raised in [8].

**Theorem 1.1.** *The set of limit points of the sequence  $\{\widehat{\mu}_\theta(n) : n \in \mathbb{Z}\}$ , with an irrational Pisot parameter  $\theta$ , is countable, so (1.1) holds for  $\nu = \mu_\theta$ .*

*Remark 1.2.* There is another way to obtain (1.1) for  $\nu = \mu_\theta$ , without getting countability of the spectrum. It is known that if a Borel probability measure  $\nu$  on  $S^1$  is  $L^p$ -improving<sup>1</sup>, then (1.1) holds, see [5, Th. 4.1]. Christ [3] proved that  $\mu_\theta$  is  $L^p$ -improving for all  $\theta > 2$ , and the case of  $\theta \in (1, 2)$  is easily reduced to that of  $\theta > 2$ . (Indeed,  $\mu_\theta$  is a convolution of  $\mu_{\theta^n}$  and some other measure, for any  $n \geq 2$ ; so if  $\mu_{\theta^n}$  is  $L^p$ -improving then  $\mu_\theta$  is  $L^p$ -improving as well.)

<sup>1</sup>A measure  $\nu$  is called  *$L^p$ -improving* for some  $p \in (1, \infty)$ , if there exists  $q = q(p) > p$  such that  $\nu * f \in L^q(S^1)$  for any  $f \in L^p(S^1)$ . If  $\nu$  improves some  $L^p$ , then it improves all of them for  $1 < p < +\infty$ , see [5].

Since we are considering measures on the circle, one may argue that it is in fact more natural to have the Bernoulli convolution measure supported on an interval of length 1, rather than on  $I_\theta$  (whose length is  $2\theta/(\theta - 1) > 1$ ). This is achieved by a linear change of variable, resulting in the Fourier coefficients  $\{\widehat{\mu}((\frac{\theta-1}{2\theta})n) : n \in \mathbb{Z}\}$ , and we show that the analog of Theorem 1.1 is still valid (see below).

More generally, one may inquire what happens under an arbitrary scale change. It turns out that the situation is rather delicate.

For  $r > 0$  put

$$\mathcal{F}_{\nu,r} := \{\widehat{\nu}(rn) : n \in \mathbb{Z}\}.$$

For a set  $E \subset \mathbb{R}$  let  $E'$  denote the derived set of  $E$ , that is, the set of its limit points.

**Theorem 1.3.** *Let  $\theta \neq 2$  be a Pisot number. Then*

- (i) *for any positive  $r \in \mathbb{Q}(\theta)$ , the set  $\mathcal{F}'_{\mu_\theta,r}$  is countably infinite;*
- (ii) *for Lebesgue a.e.  $r > 0$ , the set  $\mathcal{F}'_{\mu_\theta,r}$  is a non-empty interval.*

**Corollary 1.4.** *For any  $\theta > 2$ , the spectrum in  $L^p(S^1)$ ,  $p > 1$ , of the convolution operator corresponding to the Cantor-Lebesgue measure with the constant dissection ratio  $\theta$ , constructed on  $[0, 1) \simeq S^1 = \mathbb{R}/\mathbb{Z}$ , is countably infinite.*

This follows from Theorem 1.3 (i), since  $\frac{\theta-1}{2\theta} \in \mathbb{Q}(\theta)$  and translating the measure by  $\frac{1}{2}$  results in multiplying the Fourier coefficients by  $(-1)^n$ .

Denote  $E^{(1)} := E'$ ,  $E^{(n+1)} := (E^{(n)})'$ . For  $r \in \mathbb{Q}(\theta)$ , it is natural to ask what is the cardinality of the second, third, etc., derived sets for  $\mathcal{F}_{\mu_\theta,r}$ . The following theorem answers this question.

**Theorem 1.5.** *For any Pisot  $\theta \neq 2$  and any positive  $r \in \mathbb{Q}(\theta)$  the set  $\mathcal{F}_{\mu_\theta,r}^{(n)}$  is countably infinite for each  $n \geq 1$ .*

## 2. PROOF OF COUNTABILITY: THE MODEL CASE

The core of the paper is the proof of Theorem 1.3 (i) (which, of course, implies Theorem 1.1). Our proof is loosely based on the method used in [8] for  $\theta = 3$ . On the other hand, the case of irrational  $\theta$  requires extra tools more common for the theory of Pisot numbers (in the spirit of the monograph [2, Chapter VIII]).

The structure of the rest of the paper is as follows: in this and the next sections we are going to show that

$$(2.1) \quad \text{card } \mathcal{F}'_{\mu_\theta,r} \leq \aleph_0, \quad r \in \mathbb{Q}(\theta) \cap (0, \infty).$$

Note that our proof applies to the case  $\theta \in \mathbb{N}$  as well. In Section 4 we prove that  $\mathcal{F}_{\mu_{\theta}, r}^{(n)}$  is infinite for all  $n \geq 1$ . Combined with (2.1) this yields both Theorem 1.3 (i) and Theorem 1.5. Theorem 1.3 (ii) is also proved in Section 4.

The Pisot numbers  $\theta \neq 2$  are distinguished by the fact that  $\widehat{\mu}_{\theta}(t) \not\rightarrow 0$ , as  $t \rightarrow +\infty$  over the reals. This was proved by Erdős [4] (for integers  $\theta > 2$  this had been known earlier). It is easy to see that also  $\widehat{\mu}_{\theta}(n) \not\rightarrow 0$ , as  $n \rightarrow +\infty$  over the integers, see Section 4. If  $\theta = 2$ , then  $\mu_{\theta}$  is absolutely continuous, so there is nothing to prove.

For the rest of the paper, we fix a Pisot number  $\theta \neq 2$  and denote by  $\theta_2, \dots, \theta_m$  the conjugates of  $\theta = \theta_1$ . Since  $\theta$  is Pisot,

$$(2.2) \quad \rho = \max_{i \geq 2} |\theta_i| \in [0, 1)$$

( $\rho = 0$  if and only if  $\theta \in \mathbb{N}$ ).

Let us begin the proof of the inequality (2.1). The symbols  $\langle \cdot \rangle$  and  $\| \cdot \|$  denote the nearest integer and the distance to the nearest integer respectively.

To simplify notation, denote  $\mu := \mu_{\theta}$ . It suffices to prove that there are at most countably many limit points for the set  $\{|\widehat{\mu}(rn)| : n \in \mathbb{N}\}$ . Fix  $\eta > 0$  and assume that integers  $n_k \rightarrow +\infty$  are such that

$$(2.3) \quad |\widehat{\mu}(rn_k)| \rightarrow a, \quad a \geq \eta.$$

Similarly to [8], our goal is to show that there can be only a countable set of such  $a$ 's for any fixed  $\eta$ ; this will yield (2.1). In fact, we will show in Lemma 3.2 below that every such  $a$  satisfies

$$a = \prod_{i=0}^M \prod_{j=-\infty}^{\infty} |\cos(\pi z_i \theta^j)| \cdot \prod_{j=0}^{\infty} |\cos(2\pi r A \theta^{-j})|$$

for some  $M \geq 0$ ,  $z_i \in \mathbb{Q}(\theta)$ , and  $A \in \mathbb{Q}$ .

Given  $n_k$ , we can find  $N_k \in \mathbb{Z}$  and  $y_k \in [1, \theta)$  such that

$$(2.4) \quad y_k = 2rn_k \theta^{-N_k}.$$

Let

$$(2.5) \quad y_k \theta^j = K_j^{(k)} + \delta_j^{(k)}, \quad j = 1, \dots, N_k,$$

where  $\delta_j^{(k)} \in (-\frac{1}{2}, \frac{1}{2}]$  and  $K_j^{(k)} = \langle y_k \theta^j \rangle \in \mathbb{N}$ . By (1.2) and (2.4),

$$(2.6) \quad |\widehat{\mu}(rn_k)| = \prod_{j=-\infty}^{N_k} |\cos(\pi y_k \theta^j)| = \prod_{j=-\infty}^{N_k} \cos(\pi \delta_j^{(k)}).$$

Let  $m$  be the degree of  $\theta$ , with the minimal polynomial  $x^m - d_1x^{m-1} - \dots - d_m$ . For the rest of the paper we fix a  $\delta$  which satisfies

$$(2.7) \quad 0 < \delta < (1 + |d_1| + \dots + |d_m|)^{-1}.$$

The reason for the choice of  $\delta$  is the following

**Lemma 2.1.** *Suppose that  $|\delta_j^{(k)}| \leq \delta$  for  $j = A_k + 1, \dots, A_k + b$ , where  $0 \leq A_k \leq N_k - b$  and  $b > m$ . Then*

$$(2.8) \quad K_{j+m}^{(k)} = d_1 K_{j+m-1}^{(k)} + \dots + d_m K_j^{(k)},$$

for  $j = A_k + 1, \dots, A_k + b - m$ .

*Proof.* By our condition and (2.7), for  $j \geq A_k + 1$ ,

$$|K_{j+m}^{(k)} - d_1 K_{j+m-1}^{(k)} - \dots - d_m K_j^{(k)}| \leq \delta(1 + |d_1| + \dots + |d_m|) < 1.$$

As  $K_j^{(k)}$ 's and  $d_i$ 's are integers, we are done.  $\square$

We want to estimate the number of  $\delta_j^{(k)}$ 's that are greater than  $\delta$  in modulus. Let  $L \in \mathbb{N}$  be such that  $(\cos(\pi\delta))^L \leq \eta/2$ . It follows from (2.3) and (2.6) that for  $k$  sufficiently large,

$$(2.9) \quad L_k := \#\{j \in [1, N_k] : |\delta_j^{(k)}| > \delta\} \leq L.$$

Since we only care about the limit, we can assume without loss of generality that (2.9) holds for all  $k$ .

The rest of the proof is somewhat technical, so we believe that it is helpful first to present a sketch in the special case  $L_k = 1$  and  $r = 1$ . This will be done in the rest of the section.

Thus, let us assume for the moment that  $|\delta_j^{(k)}| \leq \delta$  for all  $j = 1, \dots, N_k$ , except possibly  $j = J_k$ . There are three possibilities: (a)  $\sup J_k < \infty$ , (b)  $\sup(N_k - J_k) < \infty$ , and (c)  $\sup J_k = \infty$  and  $\sup(N_k - J_k) = \infty$ . By passing to a subsequence, we can assume that we actually have one of the following cases:

**Case 1:**  $J_k = R$  (independent of  $k$ );

**Case 2:**  $J_k = N_k - R$ ;

**Case 3:**  $J_k \rightarrow \infty$  and  $N_k - J_k \rightarrow \infty$ .

**Case 1.** By Lemma 2.1, the sequence  $\{K_j^{(k)}\}$  satisfies the recurrence relation (2.8) for  $j = R + 1, \dots, N_k - m$  (for  $k$  large enough to satisfy  $N_k > R + m$ ). Then we can express  $K_j^{(k)}$  in terms of  $\theta$  and its conjugates  $\theta_2, \dots, \theta_m$  as follows:

$$(2.10) \quad K_j^{(k)} = c_1^{(k)} \theta^j + \sum_{i=2}^m c_i^{(k)} \theta_i^j, \quad j = R + 1, \dots, N_k.$$

Observe that the coefficients  $c_i^{(k)}$  are completely determined by  $K_j^{(k)}$  for  $j = R+1, \dots, R+m$ . These  $K_j^{(k)}$ 's are integers bounded by  $\theta^{R+m+1} + 1$  (as  $y_k \leq \theta$  and  $K_j^{(k)}$  is the nearest integer to  $y_k \theta^j$ ). Thus, there are finitely many possibilities for  $c_i^{(k)}$  and we can assume, passing to a subsequence, that  $c_i^{(k)} = c_i$  do not depend on  $k$ .

Let  $y := c_1$ ; the first important point is that  $y \in \mathbb{Q}(\theta)$ . This follows from the Cramer's Rule, solving the linear system (2.10), with  $j = R+1, \dots, R+m$ , for  $c_i$ . Alternatively, note that  $\|y\theta^j\| \rightarrow 0$  as  $j \rightarrow +\infty$  by (2.10), and the fact that  $y \in \mathbb{Q}(\theta)$  is a part of the well-known Pisot-Vijayaraghavan theorem (see [2]).

Now comes the crucial point: we have to use that  $n_k$  is an integer<sup>2</sup>. We assumed that  $r = 1$ , so  $n_k = \frac{1}{2}y_k\theta^{N_k}$  from (2.4). We have  $2n_k = K_{N_k}^{(k)}$ , since both sides are integers. Hence by (2.5) and (2.10),

$$y_k\theta^{N_k} = K_{N_k}^{(k)} = y\theta^{N_k} + O(\rho^{N_k}),$$

where  $\rho$  is given by (2.2) and the implied constant in  $O$  is independent of  $k$ . Thus,  $y_k = y + O(\theta^{-N_k}\rho^{N_k})$ , and an elementary argument yields

$$|\widehat{\mu}(n_k)| \rightarrow \prod_{j=-\infty}^{\infty} |\cos(\pi y \theta^j)|.$$

(A more general statement is proved below, in Lemma 3.2.) Since the right-hand side depends only on  $y \in \mathbb{Q}(\theta)$ , the number of possible limit points in this case is at most countable.

**Case 2.** By Lemma 2.1,  $K_j^{(k)}$ 's satisfy the recurrence relation (2.8) for  $j = 1, \dots, N_k - R - m - 1$ , when  $k$  is sufficiently large. Passing to a subsequence, we can assume that  $K_j^{(k)} = K_j$  do not depend on  $k$  for  $j \leq N_k - R - 1$  and

$$(2.11) \quad K_j = y\theta^j + \sum_{i=2}^m c_i\theta_i^j, \quad j = 1, \dots, N_k - R - 1.$$

Again we have  $y \in \mathbb{Q}(\theta)$ . Extend  $K_j$  by (2.11) to  $j = N_k - R, \dots, N_k$ ; in other words, we extend  $K_j$  to satisfy the recurrence relation (2.8). We cannot claim that  $K_j = K_j^{(k)}$  for  $j = N_k - R, \dots, N_k$ ; however, it is easy to see from the recurrence that

$$|K_j^{(k)} - K_j| \leq C_R, \quad j = N_k - R, \dots, N_k,$$

<sup>2</sup>Note that the set of limit points of  $\widehat{\mu}(t)$ , as  $t \rightarrow \infty$  over the reals, is an interval—see Lemma 4.1 below.

where  $C_R$  does not depend on  $k$ . (This is proved below, in Lemma 3.1.) Again, passing to a subsequence, we can assume that  $K_{N_k}^{(k)} - K_{N_k} = A$  is a constant. Using that  $n_k$  is an integer, we obtain

$$y_k \theta^{N_k} = K_{N_k}^{(k)} = y \theta^{N_k} + A + O(\rho^{N_k}),$$

where the implied constant in  $O$  is independent of  $k$ . Thus  $y_k = y + A\theta^{-N_k} + O(\theta^{-N_k} \rho^{N_k})$ , and it is not hard to show that

$$|\widehat{\mu}(n_k)| \rightarrow \prod_{j=-\infty}^{\infty} |\cos(\pi y \theta^j)| \cdot \prod_{j=0}^{\infty} |\cos(\pi A \theta^{-j})|.$$

(A more general statement will be proved below, in Lemma 3.2.) Since  $y \in \mathbb{Q}(\theta)$  and  $A \in \mathbb{Z}$ , the number of possible limit points in this case is again at most countable.

**Case 3.** By Lemma 2.1,  $\{K_j^{(k)}\}_{j=1}^{J_k-m-1}$  and  $\{K_j^{(k)}\}_{j=J_k+1}^{N_k-m}$  satisfy the recurrence relation (2.8). As in Case 2, we can assume by passing to a subsequence that  $K_j^{(k)} = K_j$  for  $j = 1, \dots, J_k - 1$ , whence

$$K_j = y \theta^j + \sum_{i=2}^m c_i \theta_i^j, \quad j = 1, \dots, J_k - 1,$$

for some  $y \in \mathbb{Q}(\theta)$ ,  $c_2, \dots, c_m$ . Also, as in Case 2, we extend  $K_j$  to  $j \geq J_k$  to satisfy the same recurrence relation and check that

$$|K_j^{(k)} - K_j| \leq C, \quad j = J_k + 1, \dots, J_k + m.$$

Let  $S_j^{(k)} = K_{J_k+j}^{(k)} - K_{J_k+j}$  for  $j = 1, \dots, N_k - J_k$ . Hence there exist  $b_i^{(k)} \in \mathbb{Q}(\theta_i)$ , with  $i = 1, \dots, m$ , such that

$$S_j^{(k)} = b_1^{(k)} \theta^j + \sum_{i=2}^m b_i^{(k)} \theta_i^j, \quad j = 1, \dots, m.$$

Because of the bounds on  $S_j^{(k)}$ , there are finitely many possibilities for  $b_i^{(k)}$ , so we can assume that they do not depend on  $k$ , passing to a subsequence. Let  $z = b_1 = b_1^{(k)}$ . We have  $S_j = S_j^{(k)} = z \theta^j + O(\rho^j)$  for  $j \geq 1$ . Now observe that

$$K_j^{(k)} = K_j + S_{j-J_k}, \quad j = J_k + 1, \dots, N_k.$$

Thus, using that  $n_k$  is an integer, we obtain

$$y_k \theta^{N_k} = K_{N_k}^{(k)} = K_{N_k} + S_{N_k-J_k} = y \theta^{N_k} + z \theta^{N_k-J_k} + O(\rho^{N_k-J_k}),$$

where the implied constant in  $O$  is independent of  $k$ . Since  $J_k \rightarrow \infty$  and  $N_k - J_k \rightarrow \infty$ , is not hard to show that

$$|\widehat{\mu}(n_k)| \rightarrow \prod_{j=-\infty}^{\infty} |\cos(\pi y \theta^j)| \cdot \prod_{j=-\infty}^{\infty} |\cos(\pi z \theta^j)|.$$

(We will prove a more general statement below, in Lemma 3.2.) As  $y, z \in \mathbb{Q}(\theta)$ , the number of possible limit points in this case is at most countable.

This concludes the sketch of the proof of (2.1) in the special case  $L_k = 1$  and  $r = 1$ . The idea for the general case is as follows: we gather all indices  $j$ , for which  $|\delta_j^{(k)}| > \delta$ , in groups in such a way that the distance between any two adjacent groups goes to infinity as  $k \rightarrow \infty$ . Then we treat each group similarly to one of the three cases considered in this section, depending on the position of this group (beginning, middle or end) and finally, we justify passing to the limit in the key Lemma 3.2.

### 3. PROOF OF COUNTABILITY: THE GENERAL CASE

We continue with the proof of the general case where we left it, after the definition of  $L_k$  (2.9). Let  $1 \leq I_1^{(k)} < I_2^{(k)} < \dots < I_{L_k}^{(k)} \leq N_k$  be all the indices  $j$  for which  $|\delta_j^{(k)}| > \delta$ . Since  $L_k \leq L$ , we can assume that  $L_k = L'$  does not depend on  $k$ , passing to a subsequence. Further, passing to a subsequence, we can assume that for all  $i = 1, \dots, L' - 1$ , either  $I_{i+1}^{(k)} - I_i^{(k)} = R_i$  (independent of  $k$ ), or  $I_{i+1}^{(k)} - I_i^{(k)} \rightarrow \infty$ , as  $k \rightarrow \infty$ . Also, either  $I_1^{(k)} = R_0$  or  $I_1^{(k)} \rightarrow \infty$  and either  $N_k - I_{L'}^{(k)} = R_{L'}$  or  $N_k - I_{L'}^{(k)} \rightarrow \infty$ . Let

$$R = \max\{R_i : i = 0, \dots, L'\} + 1.$$

We can find  $M \in \{1, \dots, L'\}$  and integers

$$1 = J_0^{(k)} < J_1^{(k)} < \dots < J_M^{(k)} < J_{M+1}^{(k)} = N_k$$

so that  $J_{i+1}^{(k)} - J_i^{(k)} \rightarrow \infty$  for  $i = 0, \dots, M$ , and

$$|\delta_j^{(k)}| \leq \delta \quad \text{for all } j \in \{1, \dots, N_k\} \quad \text{such that } \min_i |j - J_i^{(k)}| \geq R.$$

By Lemma 2.1,  $\{K_j^{(k)}\}$  satisfy the recurrence relation (2.8) for  $J_i^{(k)} + R \leq j \leq J_{i+1}^{(k)} - R - m$ , with  $i = 0, \dots, M$  (for  $k$  large enough to satisfy  $J_{i+1}^{(k)} - J_i^{(k)} > 2R + m$ ). In particular, this is true for  $1 + R \leq j \leq$

$J_1^{(k)} - R - m$ . Thus we can write

$$K_j^{(k)} = c_1^{(k)}\theta^j + \sum_{i=2}^m c_i^{(k)}\theta_i^j, \quad j = R+1, \dots, J_1^{(k)} - R$$

(for  $k$  sufficiently large). The coefficients  $c_i^{(k)}$  are completely determined by  $K_j^{(k)}$  for  $j = R+1, \dots, R+m$ , which are integers bounded by  $\theta^{R+m+1} + 1$ . Thus, there are finitely many possibilities for  $c_i^{(k)}$ ,  $i = 1, \dots, m$ . Hence we can assume, passing to a subsequence, that  $c_i^{(k)} = c_i$  do not depend on  $k$ . Thus,  $K_j^{(k)} = K_j$  for  $j \leq J_1^{(k)} - R$ . Denote  $z_0 := c_1$ , so that

$$K_j = z_0\theta^j + O(\rho^j).$$

As in Case 1, we have  $z_0 \in \mathbb{Q}(\theta)$  ( $z_0$  is a natural analog of  $y$  from the previous section). Next we repeat the argument from Case 3. Extend the sequence  $\{K_j\}$  to  $j > J_1^{(k)} - R$  so that it satisfies the recurrence relation for all  $j$ . We need the following lemma.

**Lemma 3.1.** *Suppose that  $K_j^{(k)} = K_j$  for  $j \leq A_k$  and  $\{K_j\}$  satisfies the recurrence relation (2.8) for all  $j$ . Then for any  $p \in \mathbb{N}$  there exists  $C_p = C_p(\theta) > 0$  such that*

$$(3.1) \quad |K_j^{(k)} - K_j| \leq C_p, \quad j = A_k + 1, \dots, A_k + p.$$

*Proof.* This is proved by induction. For any  $j \geq 1$ , since  $K_j^{(k)}$  is the nearest integer to  $y_k\theta^j$ ,

$$|K_{j+m}^{(k)} - d_1K_{j+m-1}^{(k)} - \dots - d_mK_j^{(k)}| \leq \frac{1}{2}(1 + |d_1| + \dots + |d_m|) =: \gamma.$$

It follows that we can take  $C_1 = \gamma$  in (3.1). Suppose (3.1) is verified for some  $p$ . Then we have for  $j = A_k + p + 1$ ,

$$\begin{aligned} |K_{j+1}^{(k)} - K_{j+1}| &= |K_{j+1}^{(k)} - d_1K_j^{(k)} - \dots - d_mK_{j-m+1}^{(k)} \\ &\quad + d_1K_j^{(k)} + \dots + d_mK_{j-m+1}^{(k)} - d_1K_j - \dots - d_mK_{j-m+1}| \\ &\leq \gamma + (|d_1| + \dots + |d_m|)C_p < \gamma(1 + 2C_p). \end{aligned}$$

Thus, we may put  $C_{p+1} = \gamma(1 + 2C_p)$ , and the lemma is proved.  $\square$

Let

$$S_j^{(k)} = K_{J_1^{(k)}+j}^{(k)} - K_{J_1^{(k)}+j}, \quad j = R+1, \dots, R+m.$$

By Lemma 3.1,

$$|S_j^{(k)}| \leq C_{2R+m}, \quad j = R+1, \dots, R+m.$$

Therefore, we can assume (passing to a subsequence) that  $S_j^{(k)}$ 's do not depend on  $k$  for  $j = R+1, \dots, R+m$ . We can find  $z_1 \in \mathbb{Q}(\theta), c'_2, \dots, c'_m$  so that

$$(3.2) \quad S_j = S_j^{(k)} = z_1 \theta^j + \sum_{i=2}^m c'_i \theta_i^j, \quad j = R+1, \dots, R+m.$$

Extend  $S_j$  to  $j > R+m$  by the formula (3.2), so that they satisfy the recurrence relation. Now observe that

$$K_j^{(k)} = K_j + S_{j-J_1^{(k)}}, \quad j = J_1^{(k)} + R+1, \dots, J_2^{(k)} - R$$

(for  $k$  large enough to satisfy  $J_2^{(k)} - J_1^{(k)} > 2R+m$ ), as both sides agree for  $j = J_1^{(k)} + R+1, \dots, J_1^{(k)} + R+m$ , and satisfy the same recurrence relation of length  $m$ . It follows that

$$K_j^{(k)} = z_0 \theta^j + z_1 \theta^{j-J_1^{(k)}} + O(\rho^{j-J_1^{(k)}}), \quad j = J_1^{(k)} + R+1, \dots, J_2^{(k)} - R,$$

where the implied constant in  $O$  is independent of  $k$ .

Next we repeat the same argument and obtain, by induction, that for  $i = 2, \dots, M$ ,

$$(3.3) \quad \begin{aligned} K_j^{(k)} &= z_0 \theta^j + z_1 \theta^{j-J_1^{(k)}} + \dots + z_i \theta^{j-J_i^{(k)}} + O(\rho^{j-J_i^{(k)}}), \\ j &= J_i^{(k)} + R, \dots, J_{i+1}^{(k)} - R. \end{aligned}$$

Indeed, for each  $i$  extend  $K_j^{(k)}$  from  $j < J_i^{(k)} - R$  to larger  $j$ 's by recurrence; denote them as  $Q_j^{(k)}$ . Put

$$T_j^{(k)} := K_{J_i^{(k)}+j}^{(k)} - Q_{J_i^{(k)}+j}^{(k)}, \quad j = R+1, \dots, R+m.$$

Then  $|T_j^{(k)}| \leq C_{2R+m}$  for  $j = R+1, \dots, R+m$ . We can write  $T_j^{(k)}$  as a linear combination of  $\theta^j$  and the powers of its conjugates. As above, there are finitely many possibilities for the coefficients (as  $k$  varies), so we can assume without loss of generality that they do not depend on  $k$ . The coefficient at  $\theta^j$  will be denoted by  $z_i$ , which yields (3.3).

For  $i = M$  the formula (3.3) becomes

$$(3.4) \quad \begin{aligned} K_{N_k-R+j}^{(k)} &= z_0 \theta^{N_k-R+j} + \sum_{i=1}^M z_i \theta^{N_k-J_i^{(k)}-R+j} + O(\rho^{N_k-J_M^{(k)}-R+j}), \\ j &= J_M^{(k)} + 2R - N_k, \dots, 0 \end{aligned}$$

(recall that  $J_M^{(k)} - N_k \rightarrow -\infty$ ). As usual, the implied constant in  $O$  is independent of  $k$ . One last time extend  $K_j^{(k)}$  by recurrence, to

$j = N_k - R + 1, \dots, N_k, \dots$ . Denote the resulting integer sequence by  $\{L_{N_k - R + j}^{(k)}\}_{j=1}^\infty$ . By Lemma 3.1,

$$(3.5) \quad |K_{N_k}^{(k)} - L_{N_k}^{(k)}| \leq C_R.$$

We have

$$(3.6) \quad 2rn_k = y_k \theta^{N_k} = K_{N_k}^{(k)} + \delta_{N_k}^{(k)}.$$

Now we use the fact that  $n_k$  is an integer; this is slightly more complicated than in Section 2, because the left-hand side of (3.6) need not be an integer. However, as  $2r \in \mathbb{Q}(\theta) \setminus \{0\}$  by assumption, we can invert it in  $\mathbb{Q}(\theta)$ , i.e.,

$$(3.7) \quad (2r)^{-1} = a_0 + a_1\theta + \dots + a_{m-1}\theta^{m-1},$$

for some  $a_i \in \mathbb{Q}$ . Thus,

$$(3.8) \quad \begin{aligned} n_k &= (2r)^{-1}(K_{N_k}^{(k)} + \delta_{N_k}^{(k)}) \\ &= (a_0 + a_1\theta + \dots + a_{m-1}\theta^{m-1})K_{N_k}^{(k)} + (2r)^{-1}\delta_{N_k}^{(k)} \\ &= a_0L_{N_k}^{(k)} + a_1L_{N_k+1}^{(k)} + \dots + a_{m-1}L_{N_k+m-1}^{(k)} + A_k. \end{aligned}$$

Let us estimate the error term  $A_k$ . By definition, the integers  $L_{N_k - R + j}^{(k)}$  satisfy (3.4), with  $K$  replaced by  $L$ , for  $j = 1, 2, \dots$ . In particular,

$$(3.9) \quad L_{N_k+j}^{(k)} = z_0\theta^{N_k+j} + \sum_{i=1}^M z_i\theta^{N_k - J_i^{(k)} + j} + O(\rho^{N_k - J_M^{(k)} + j}), \quad j \geq 0.$$

Hence in view of  $N_k - J_M^{(k)} \rightarrow +\infty$ ,

$$\lim_{k \rightarrow \infty} |L_{N_k+j+1}^{(k)} - \theta L_{N_k+j}^{(k)}| = 0, \quad 0 \leq j \leq m-1,$$

whence

$$\left| L_{N_k}^{(k)} \sum_{j=0}^{m-1} a_j \theta^j - \sum_{j=0}^{m-1} a_j L_{N_k+j}^{(k)} \right| \rightarrow 0, \quad k \rightarrow +\infty.$$

By (3.8), (3.5) and in view of  $|\delta_{N_k}^{(k)}| \leq \frac{1}{2}$ , we have  $|A_k| \leq C'$  for some constant  $C'$  independent of  $k$ ; namely, one may put for  $k$  large enough,

$$C' = 1 + (4r)^{-1}(2C_R + 1),$$

where  $C_R$  is as in (3.5).

On the other hand, it follows from (3.8) that  $A_k \in s^{-1}\mathbb{Z}$  for some  $s \in \mathbb{N}$  independent of  $k$ , because  $a_i \in \mathbb{Q}$  and the  $n_k$  is an integer. Thus,

there are finitely many possibilities for  $A_k$ , so, passing to a subsequence, we can assume that  $A_k = A$  is a constant. Now we have by (3.9),

$$\begin{aligned} y_k &= 2rn_k\theta^{-N_k} = 2r(a_0L_{N_k}^{(k)} + \cdots + a_{m-1}L_{N_k+m-1}^{(k)})\theta^{-N_k} + 2rA\theta^{-N_k} \\ &= 2rz_0(a_0 + a_1\theta + \cdots + a_{m-1}\theta^{m-1}) \\ &\quad + 2r\sum_{i=1}^M z_i(a_0\theta^{-J_i^{(k)}} + \cdots + a_{m-1}\theta^{-J_i^{(k)}+m-1}) + 2rA\theta^{-N_k} \\ &\quad + O(\theta^{-N_k}\rho^{N_k-J_M^{(k)}}), \end{aligned}$$

and finally, by (3.7),

$$(3.10) \quad y_k = z_0 + \sum_{i=1}^M z_i\theta^{-J_i^{(k)}} + 2rA\theta^{-N_k} + O(\theta^{-N_k}\rho^{N_k-J_M^{(k)}}).$$

**Lemma 3.2.** *We have*

$$\begin{aligned} |\widehat{\mu}(n_k)| &= \prod_{j=-\infty}^{N_k} |\cos(\pi y_k \theta^j)| \\ &\rightarrow \prod_{i=0}^M \prod_{j=-\infty}^{\infty} |\cos(\pi z_i \theta^j)| \cdot \prod_{j=0}^{\infty} |\cos(2\pi r A \theta^{-j})|, \quad k \rightarrow +\infty. \end{aligned}$$

Recall that by the construction of  $z_i$  we have for  $0 \leq i \leq M$ :

$$(3.11) \quad \|z_i \theta^j\| = O(\rho^j), \quad j \rightarrow \infty.$$

It follows that the bi-infinite products in the right-hand side converge. This lemma will clearly imply (2.1), as  $z_i \in \mathbb{Q}(\theta)$  and  $A \in \mathbb{Q}$ , so there are at most countably many possible limits.

*Proof of Lemma 3.2.* We can find integers  $E_i^{(k)}$ ,  $i = 0, \dots, M$ , so that

$$J_i^{(k)} < E_i^{(k)} < J_{i+1}^{(k)} \quad \text{and} \quad \lim_{k \rightarrow \infty} (E_i^{(k)} - J_i^{(k)}) = \lim_{k \rightarrow \infty} (J_{i+1}^{(k)} - E_i^{(k)}) = +\infty.$$

We are going to show that

(3.12)

$$F_0(k) := \frac{\prod_{j=-\infty}^{E_0^{(k)}} |\cos(\pi y_k \theta^j)|}{\prod_{j=-\infty}^{\infty} |\cos(\pi z_0 \theta^j)|} \rightarrow 1, \quad k \rightarrow \infty;$$

(3.13)

$$F_i(k) := \frac{\prod_{j=E_{i-1}^{(k)}+1}^{E_i^{(k)}} |\cos(\pi y_k \theta^j)|}{\prod_{j=-\infty}^{\infty} |\cos(\pi z_i \theta^j)|} \rightarrow 1, \quad k \rightarrow \infty, \quad i = 1, \dots, M;$$

(3.14)

$$F_{M+1}(k) := \frac{\prod_{j=E_M^{(k)}+1}^{N_k} |\cos(\pi y_k \theta^j)|}{\prod_{j=0}^{\infty} |\cos(2\pi r A \theta^{-j})|} \rightarrow 1, \quad k \rightarrow \infty.$$

These statements will imply the lemma.

First we verify (3.12). Observe that  $\prod_{j=E_0^{(k)}}^{\infty} |\cos(\pi z_0 \theta^j)| \rightarrow 1$ , since  $E_0^{(k)} \rightarrow \infty$  and the denominator in  $F_0(k)$  converges. Thus, it remains to show that

$$\prod_{j=-\infty}^{E_0^{(k)}} \frac{|\cos(\pi y_k \theta^j)|}{|\cos(\pi z_0 \theta^j)|} \rightarrow 1.$$

Note that  $y_k = z_0 + O(\theta^{-J_1^{(k)}})$  by (3.10). By assumption (2.3),  $|\cos(\pi y_k \theta^j)| \geq \eta$  for  $j \leq N_k$ . Since  $|y_k \theta^j - z_0 \theta^j| = O(\theta^{j-J_1^{(k)}})$ , we have  $|\cos(\pi y' \theta^j)| \geq \eta/2$  for  $y'$  between  $z_0$  and  $y_k$ , for all  $j \leq E_0^{(k)}$ , for  $k$  sufficiently large. Then we can take the logarithm of each term and use the mean value theorem to get

$$\begin{aligned} \left| \log \left| \frac{\cos(\pi y_k \theta^j)}{\cos(\pi z_0 \theta^j)} \right| \right| &\leq |\tan(\pi y' \theta^j)| \cdot |y_k - z_0| \theta^j \\ &\leq \frac{2}{\eta} |y_k - z_0| \theta^j = O(\theta^{j-J_1^{(k)}}), \end{aligned}$$

where  $y'$  is between  $z_0$  and  $y_k$ . Summing over  $j \leq E_0^{(k)}$  and letting  $k \rightarrow \infty$  yields the desired claim, since  $J_1^{(k)} - E_0^{(k)} \rightarrow +\infty$ .

Now we verify (3.13). Since  $E_{i-1}^{(k)} - J_i^{(k)} \rightarrow -\infty$ ,  $E_i^{(k)} - J_i^{(k)} \rightarrow +\infty$ , and the denominator in  $F_i(k)$  converges, it suffices to show that

$$(3.15) \quad \prod_{j=E_{i-1}^{(k)}+1}^{E_i^{(k)}} \frac{|\cos(\pi y_k \theta^j)|}{|\cos(\pi z_i \theta^{j-J_i^{(k)}})|} \rightarrow 1.$$

In view of (3.10), we can write

$$y_k \theta^j = z_0 \theta^j + \sum_{\ell=1}^{i-1} z_\ell \theta^{j-J_\ell^{(k)}} + z_i \theta^{j-J_i^{(k)}} + O(\theta^{j-J_{i+1}^{(k)}}),$$

for  $j = E_{i-1}^{(k)} + 1, \dots, E_i^{(k)}$ . By (3.11), for  $j = E_{i-1}^{(k)} + 1, \dots, E_i^{(k)}$  we have

$$y_k \theta^j = O(\rho^{j-J_{i-1}^{(k)}}) + z_i \theta^{j-J_i^{(k)}} + O(\theta^{j-J_{i+1}^{(k)}}) \pmod{\mathbb{Z}}.$$

For  $k$  sufficiently large, the denominators in (3.15) are bounded away from 0, as above, when we checked (3.12), and we obtain

$$\left| \log \left| \frac{\cos(\pi y_k \theta^j)}{\cos(\pi z_i \theta^{j-J_i^{(k)}})} \right| \right| \leq \text{const} \cdot (\rho^{j-J_{i-1}^{(k)}} + \theta^{j-J_{i+1}^{(k)}}).$$

Summing over  $j = E_{i-1}^{(k)} + 1, \dots, E_i^{(k)}$ , we obtain that the logarithm of the product in (3.15) is bounded in modulus by  $\text{const} \cdot (\rho^{E_{i-1}^{(k)}-J_{i-1}^{(k)}} + \theta^{E_i^{(k)}-J_{i+1}^{(k)}})$  which tends to 0, as  $k \rightarrow \infty$ .

It remains to check (3.14). Since  $E_M^{(k)} - N_k \rightarrow -\infty$  and the denominator in  $F_{M+1}(k)$  converges, it is sufficient to show that

$$(3.16) \quad \prod_{j=E_M^{(k)}+1}^{N_k} \frac{|\cos(\pi y_k \theta^j)|}{|\cos(2\pi r A \theta^{j-N_k})|} \rightarrow 1.$$

We have for  $j = E_M^{(k)} + 1, \dots, N_k$ , from (3.10), in view of (3.11):

$$y_k \theta^j = 2r A \theta^{j-N_k} + O(\rho^{j-J_M^{(k)}}) + O(\theta^{j-N_k} \rho^{N_k-J_M^{(k)}}) \pmod{\mathbb{Z}}.$$

For  $k$  sufficiently large, the denominators in (3.16) are bounded away from 0, as above, when we checked (3.12), and we can write

$$\left| \log \left| \frac{\cos(\pi y_k \theta^j)}{\cos(2\pi r A \theta^{j-N_k})} \right| \right| \leq \text{const} \cdot (\rho^{j-J_M^{(k)}} + \theta^{j-N_k} \rho^{N_k-J_M^{(k)}}).$$

Summing over  $j = E_M^{(k)} + 1, \dots, N_k$ , we obtain that the logarithm of the product in (3.16) is bounded above in modulus by  $\text{const} \cdot (\rho^{E_M^{(k)}-J_M^{(k)}} + \rho^{N_k-J_M^{(k)}})$  which tends to 0, as  $k \rightarrow \infty$ . This concludes the proof of Lemma 3.2 and inequality (2.1).  $\square$

#### 4. PROOFS OF OTHER RESULTS

**4.1. Proof of Theorem 1.3 (ii).** Let  $J_\theta$  denote the set of limit points of  $\{\widehat{\mu}_\theta(t) : t > 0\}$  as  $t \rightarrow \infty$ .

**Lemma 4.1.**  *$J_\theta$  is a non-empty interval.*

*Proof.* By the result of Erdős [4],  $J_\theta$  contains a nonzero point. On the other hand,  $0 \in J_\theta$ , since  $\widehat{\mu}_\theta(\theta^n/4) = 0$ ,  $n \geq 1$ . Let  $a = \inf J_\theta$ ,  $b = \sup J_\theta$ . Then there are sequences  $u_i, v_i \rightarrow \infty$  such that  $\widehat{\mu}_\theta(u_i) \rightarrow a$  and  $\widehat{\mu}_\theta(v_i) \rightarrow b$ . Without loss of generality,  $u_i < v_i < u_{i+1}$  for all  $i$ . Since  $t \mapsto \widehat{\mu}_\theta(t)$  is continuous, for any  $\varepsilon > 0$ , for all  $i$  sufficiently large, any value between  $a + \varepsilon$  and  $b - \varepsilon$  is assumed by  $\widehat{\mu}_\theta(t)$  at least once in  $(u_i, v_i)$ . Thus,  $J_\theta = [a, b]$ .  $\square$

Our goal is to prove that for a.e.  $r > 0$ ,  $J_\theta$  is in fact the set of limit points for the sequence  $\{\widehat{\mu}_\theta(rn) : n \in \mathbb{Z}\}$  as well.

Let  $\{y_k\}_{k \geq 1}$  be a sequence dense in  $J_\theta$ . By the definition of  $J_\theta$ , for any  $k \geq 1$ , there is a sequence  $t_i^{(k)} \rightarrow +\infty$  as  $i \rightarrow \infty$ , such that  $\lim_{i \rightarrow \infty} \widehat{\mu}_\theta(t_i^{(k)}) = y_k$ . Recall the following well-known fact.

**Proposition 4.2.** [6, Chap. 1, Sec. 4, Cor 4.3] *For any unbounded sequence  $\{x_i\}_{i \geq 1}$ , the set  $\{\alpha x_i\}_{i \geq 1}$  is dense modulo 1 for Lebesgue-a.e.  $\alpha$ .*

Thus, for a.e.  $r > 0$ , the sequence  $\{r^{-1}t_i^{(k)}\}_{i \geq 1}$  is dense modulo 1 for all  $k \geq 1$ . Fix such an  $r$ . Then for any  $k \geq 1$ , there is a subsequence  $t_{i_j}^{(k)}$  such that  $r^{-1}t_{i_j}^{(k)} \rightarrow 0 \pmod{1}$  (of course,  $i_j$  may depend on  $k$ ). Thus, for any  $k$ , there exist  $n_j \in \mathbb{N}$  such that  $rn_j - t_{i_j}^{(k)} \rightarrow 0$ , as  $j \rightarrow \infty$ . Note that  $|\frac{d}{dt}\widehat{\mu}_\theta(t)| \leq C$ , since  $\mu_\theta$  has compact support on  $\mathbb{R}$ , whence

$$(4.1) \quad \{\widehat{\mu}_\theta(a_n) : n \in \mathbb{N}\}' = \{\widehat{\mu}_\theta(b_n) : n \in \mathbb{N}\}'$$

for any  $a_n, b_n \rightarrow \infty$ , with  $a_n - b_n \rightarrow 0$ . Therefore,

$$\lim_{j \rightarrow \infty} \widehat{\mu}_\theta(rn_j) = \lim_{j \rightarrow \infty} \widehat{\mu}_\theta(t_{i_j}^{(k)}) = y_k.$$

It follows that for a.e.  $r > 0$ , the set of limit points of  $\{\widehat{\mu}_\theta(rn) : n \in \mathbb{N}\}$  contains all  $y_k$ , which are dense in  $J_\theta$ , and hence it contains all of  $J_\theta$ .  $\square$

#### 4.2. Conclusion of the proof of Theorem 1.3 (i) and proof of

**Theorem 1.5.** In view of (2.1) and the fact that  $\mathcal{F}_{\mu_\theta, r}^{(n+1)} \subset \mathcal{F}_{\mu_\theta, r}^{(n)}$ ,  $n \geq 1$ , it suffices to show that the  $n$ th derived set of limit points of  $\mathcal{F}_{\mu_\theta, r}$  is infinite for  $r \in \mathbb{Q}(\theta)$ .

Since  $r \in \mathbb{Q}(\theta)$ , there exists  $p \in \mathbb{Z}[x]$  such that  $\Lambda = rp(\theta) \in \mathbb{Z}[\theta]$ . Let  $q \in \mathbb{Z}[\theta]$  be an arbitrary number. We have

$$\lim_{k \rightarrow \infty} \widehat{\mu}_\theta(r\langle p(\theta)q\theta^k \rangle) = \lim_{k \rightarrow \infty} \widehat{\mu}_\theta(\Lambda q\theta^k) = \prod_{j=-\infty}^{\infty} \cos(2\pi \Lambda q\theta^j).$$

The first equality holds by (4.1), as  $\|h\theta^n\| = O(\rho^n) = o(1)$  for any  $h \in \mathbb{Z}[\theta]$ . The second equality follows from (1.2). Put

$$\varphi_\Lambda(q) = \prod_{j=-\infty}^{\infty} \cos(2\pi\Lambda q\theta^j)$$

and

$$\Omega_\Lambda := \{\varphi_\Lambda(q) : q \in \mathbb{Z}[\theta]\}.$$

We just proved that  $\Omega_\Lambda \subset \mathcal{F}'_{\mu_\theta, r}$ , so  $\Omega_\Lambda^{(n)} \subset \mathcal{F}'_{\mu_\theta, r}^{(n+1)}$ ,  $n \geq 1$ . Our next goal is to show that  $\Omega'_\Lambda$  (and hence  $\mathcal{F}''_{\mu_\theta, r}$ ) is infinite for every  $\Lambda \in \mathbb{Z}[\theta]$ . Let  $a, b \in \mathbb{Z}[\theta]$  and put  $q_n(a, b) := a + b\theta^n$ . Then

$$\begin{aligned} \varphi_\Lambda(q_n(a, b)) &= \prod_{j=-\infty}^{\lfloor \frac{n}{2} \rfloor} \cos(2\pi\Lambda(a + b\theta^n)\theta^{-j}) \cdot \prod_{j=\lfloor \frac{n}{2} \rfloor + 1}^{\lfloor \frac{3n}{2} \rfloor} \cos(2\pi\Lambda(a + b\theta^n)\theta^{-j}) \\ &\quad \cdot \prod_{j=\lfloor \frac{3n}{2} \rfloor + 1}^{\infty} \cos(2\pi\Lambda(a + b\theta^n)\theta^{-j}), \end{aligned}$$

and similarly to Lemma 3.2, it is easy to see that the first product tends to  $\varphi_\Lambda(a)$ , the second one tends to  $\varphi_\Lambda(b)$  and the last one tends to 1. Hence

$$(4.2) \quad \varphi_\Lambda(q_n(a, b)) \rightarrow \varphi_\Lambda(a)\varphi_\Lambda(b), \quad n \rightarrow +\infty.$$

Recall that  $\theta \neq 2$  (when  $\varphi_\Lambda(q) \equiv 0$  for any  $\Lambda$  and  $q$ ), so there always exists  $q$  such that  $0 < |\varphi_\Lambda(q)| < 1$ . Furthermore, since  $q_n(a, b) \in \mathbb{Z}[\theta]$ , (4.2) implies that  $\Omega_\Lambda^2 \subset \Omega'_\Lambda$ , where  $\Omega_\Lambda^2 = \{\omega_1\omega_2 : \omega_i \in \Omega_\Lambda\}$ . Now,  $\Omega_\Lambda^4 \subset (\Omega'_\Lambda)^2 \subset (\Omega_\Lambda^2)' \subset \Omega''_\Lambda$ , and by induction,  $\Omega_\Lambda^{2^n} \subset \Omega_\Lambda^{(n)}$ . Thus,  $\varphi_\Lambda^n(q) \in \Omega_\Lambda^{(n)}$ , and since the derived sets are nested, they are all infinite. It follows that  $\mathcal{F}_{\mu_\theta, r}^{(n)}$  is countably infinite for each  $n \geq 1$ .  $\square$

## 5. CONCLUDING REMARKS

**5.1.** Our first remark concerns the proof of Theorem 1.3 (ii) (see the beginning of the previous section). In fact, what we use is the following

**Lemma 5.1.** *Assume that  $f$  is uniformly continuous on  $\mathbb{R}_+$  and  $J$  is the set of the limit points of  $f$  as  $t \rightarrow +\infty$ . Then the derived set for  $\{f(rn) : n \in \mathbb{N}\}$  as  $n \rightarrow +\infty$  is equal to  $J$  for a.e.  $r > 0$ .*

The proof of this lemma is exactly the same as above for  $f(t) := \widehat{\mu}_\theta(t)$ . This claim is probably known but we did not find it in the literature.

**5.2.** Our second remark consists in a simple observation that the expression for the limit points of  $\mathcal{F}_{\mu_\theta, r}$  in Lemma 3.2 (without the moduli) is in fact a general formula for  $x \in \mathcal{F}'_{\mu_\theta, r}$ . More precisely, let

$$\mathcal{P}_\theta = \{\xi : \|\xi\theta^n\| \rightarrow 0, n \rightarrow +\infty\}.$$

As is well known,  $\mathbb{Z}[\theta] \subsetneq \mathcal{P}_\theta \subsetneq \mathbb{Q}(\theta)$  (see, e.g., [2]), and  $\mathcal{P}_\theta$  is obviously a group under addition. Then our claim is that for any  $M \in \mathbb{Z}_+$ ,  $(z_i)_{i=0}^M \in \mathcal{P}_\theta^{M+1}$  and  $A \in \mathbb{Z}$ ,

$$x = \prod_{i=0}^M \prod_{j=-\infty}^{\infty} \cos(\pi z_i \theta^j) \cdot \prod_{j=0}^{\infty} \cos(2\pi r A \theta^{-j}) \in \mathcal{F}'_{\mu_\theta, r}.$$

Indeed, put

$$n_k = \langle (2r)^{-1}(z_0\theta^{(M+1)k} + z_1\theta^{Mk} + \dots + z_M\theta^k) \rangle + A.$$

The proof is practically the same as for Lemma 3.2, and we leave it to the reader.

**5.3.** Our next remark concerns translations of Bernoulli convolutions. Let  $\gamma \in \mathbb{R}$ ; then shifting the origin by  $\gamma$  results in multiplying  $\widehat{\mu}_\theta(rn)$  by  $e^{2\pi i \gamma n}$ .

**Proposition 5.2.** *The closure of the set  $\{\widehat{\mu}_\theta(rn)e^{2\pi i \gamma n} : n \in \mathbb{Z}\}$  is countable for  $r \in \mathbb{Q}(\theta)$  and  $\gamma \in \mathbb{Q}(\theta)$ , but uncountable for  $r \in \mathbb{Q}(\theta)$  and a.e.  $\gamma \in \mathbb{R}$ .*

*Proof.* First suppose that  $r \in \mathbb{Q}(\theta)$  and  $\gamma \in \mathbb{Q}(\theta)$ . If  $\widehat{\mu}_\theta(rn_k)e^{2\pi i \gamma n_k}$  has a nonzero limit, then  $|\widehat{\mu}_\theta(rn_k)| \rightarrow a \neq 0$  and  $\|\gamma n_k\| \rightarrow b \in [0, \frac{1}{2}]$ . By Theorem 1.3 (i), there are countably many possible  $a$ 's, so it remains to check that there are at most countably many possible  $b$ 's. We can proceed as in Section 3 and obtain (3.10), without loss of generality. This implies that  $n_k$  is approximated by a finite linear combination of powers of  $\theta$  with coefficients in  $\mathbb{Q}(\theta)$ . Now it is enough to note that for any  $\xi \in \mathbb{Q}(\theta)$ , the sequence  $\{\|\xi\theta^n\| : n \in \mathbb{N}\}$  has finitely many limit points<sup>3</sup> and hence the sequence  $\{\|\gamma n_k\| : k \in \mathbb{N}\}$  has finitely many limit points.

For the proof of the second statement, let  $r \in \mathbb{Q}(\theta)$  and fix  $n_k$  so that  $|\widehat{\mu}_\theta(rn_k)| \rightarrow a \neq 0$ . Using Proposition 4.2 again, we see that for a.e.  $\gamma \in \mathbb{R}$ , the sequence  $\{\gamma n_k : k \in \mathbb{N}\}$  is dense modulo 1, and therefore, the set of limit points of the sequence  $\{\widehat{\mu}_\theta(rn_k)e^{2\pi i \gamma n_k}\}$  is the circle of radius  $|a|$ .  $\square$

<sup>3</sup>This was known to Pisot, see e.g. [1, p. 96]

**5.4.** Denote by  $\mu_\theta(r, \gamma)$  the measure on the circle whose Fourier coefficients are  $\widehat{\mu}_\theta(rn)e^{2\pi in\gamma}$  (that is, the translation of a scaled copy of  $\mu_\theta$ ). We have shown that for most  $(r, \gamma)$  the spectrum  $\text{sp}(T_{\mu_\theta(r, \gamma)}, L^2)$  contains a continuum (an interval or a circle). In these cases we cannot use Sarnak's result [8] to claim that the spectra are the same in all  $L^p$ , for  $p \in (1, \infty)$ . However, the remarks in [3] indicate that  $\mu_\theta(r, 0)$  is  $L^p$ -improving for any  $r > 1$  and  $\theta > 1$ , and any translation of the measure, obviously, preserves the property. Thus, by [5, Th. 4.1] we see that the claim on the coincidence of spectra in all  $L^p$  is valid for all  $\mu_\theta(r, \gamma)$ .

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