Representations of General Linear Lie Superalgebras

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The general linear Lie algebra \( \mathfrak{gl}_n(\mathbb{C}) \) --- endomorphisms of an \( n \)-dimensional complex vector space \( V \) with operation \([x,y] = xy - yx\) being the commutator --- is the most basic example of a Lie algebra. If we assume instead that the vector space \( V \) is equipped with a \( \mathbb{Z}/2 \)-grading, i.e. \( V = V_0 \oplus V_1 \) is the direct sum of an \( m \)-dimensional "even" and an \( n \)-dimensional "odd" subspace, and replace commutator with the "supercommutator" (which takes account of parity in the most natural way), we get the general linear Lie superalgebra \( \mathfrak{gl}_{m|n}(\mathbb{C}) \).

The representation theory of the general linear Lie algebra is unbelievably rich and has a long history going back to Schur and Weyl, who computed the characters of the finite dimensional irreducible representations via the theory of symmetric functions. There is also an important family of (mostly infinite dimensional) irreducible highest weight representations for which an explicit character formula was conjectured by Kazhdan and Lusztig in 1979, and dramatically proved by Beilinson-Bernstein and Brylinski-Kashiwara in 1980, thereby giving birth to a subject known as "geometric representation theory."

So what happens for the general linear Lie superalgebra? Even the finite dimensional representations are quite difficult to understand, but now we have a pretty good picture. There is also a version of the Kazhdan-Lusztig conjecture which has just been proved (by Cheng, Lam and Wang). In the talk I'll try to give you the flavor of these results, starting with the base cases \( \mathfrak{gl}_2(\mathbb{C}) \) and \( \mathfrak{gl}_{1|1}(\mathbb{C}) \), which illustrate the general picture perfectly despite being trivial from a combinatorial perspective.