Instructions: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. Please start each solution on a new page and submit your solutions in order. Some problems have more than one part; the parts may not be weighted equally.

You may use any standard theorem from your complex analysis text, identifying it either by name or by stating it in full. Be sure to establish that the hypotheses of the theorem are satisfied before you use it.

$\mathbb{R}$ is the real line, $\mathbb{C}$ is the complex plane, $\mathbb{D}$ is the open unit disc $\{z : |z| < 1\}$.

1. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(xt)}{2+2x+x^2}\,dx$ for $x \in \mathbb{R}$.

2. Let $\{f_n\}$ be a sequence of analytic one-to-one functions on a connected open set $G \subset \mathbb{C}$. Suppose that $f_n \to f$ uniformly on compact subsets of G. Show that f is analytic on G, and that it is either one-to-one or constant on G.

3. Prove that the function $f(z) = \sin(z) - z^2$ has infinitely many zeros in $\mathbb{C}$.

4. Suppose f is entire and $|f(z)| = 1$ for all $z \in \mathbb{R}$. Prove that $f(z) = e^{g(z)}$ where g is entire.

5. Suppose $f(z)$ is a non-constant entire function which satisfies, for all z,
$$f(z+1) = f(z) \quad \text{and} \quad f(z+i) = e^{-4\pi i z} f(z).$$
How many zeroes does f have in the set $\{z : 0 \le \mathrm{Re}(z) < 1,\ 0 \le \mathrm{Im}(z) < 1\}$?

6. Let K be a countable closed subset of $\mathbb{C}$. Prove that if f is bounded and analytic on $\mathbb{C}\backslash K$ then f is constant.

7. Prove that any function f which is analytic in $\mathbb{C}\backslash\big([0,1] \cup [2,3]\big)$ can be written as $f = g_1 + g_2$, where g_1 is analytic in $\mathbb{C}\backslash[0,1]$ and g_2 is analytic in $\mathbb{C}\backslash[2,3]$.

8. For $j = 1, 2$, let $f_j(z)$ be a 1-1 analytic map from $\mathbb{D}$ onto G_j. Assume that $f_j(0) = 0$ for $j = 1, 2$, and that $G_1 \subset G_2$. Show that $|f_1'(0)| \le |f_2'(0)|$.

   Give an example to show that the result can fail if f_2 is not 1-1.