Algebra prelim 2017

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

1. Let R be a Noetherian ring. Prove that $R[x]$ and $R[\![x]\!]$ are both Noetherian. (The first part of the question is asking you to prove the Hilbert Basis Theorem, not to use it!)

2. Classify (with proof) all fields with finitely many elements.

3. Suppose A is a commutative ring and M is a finitely presented module. Given any surjection $\phi : A^n \to M$ from a finite free A-module, show that $\ker(\phi)$ is finitely generated.

4. Classify all groups of order 57.

5. Let H_n denote the Heisenberg group of order n^3, i.e.,

$$H_n \; = \; \langle a, b, c \mid a^n = b^n = c^n = 1,\ ba = cab, c \text{ is central}\rangle.$$

   (a) Show that

$$H_n \;\cong\; \left\{ \begin{pmatrix} 1 & u & w \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; u, v, w \in \mathbb{Z}/n\mathbb{Z} \right\}$$

   (b) Classify the irreducible complex representations of H_p when p is prime.

6. Find all ring homomorphisms $\mathbb{Q}[x]/(x^{100}+2) \to \mathbb{Q}[x]/(x^{1001}+2)$.

7. Show that a finite simple group can not have a 2-dimensional irreducible representation over $\mathbb{C}$. (Hint: the determinant might prove useful.)

8. If q is a power of a prime let $\mathbb{F}_q$ denote the field with q elements. Given an explicit description of $M_2(\mathbb{F}_4) \otimes_{\mathbb{F}_2} M_2(\mathbb{F}_4)$ as a direct product of matrix rings over division rings.