2020 Spring Algebra Prelim
March, 2020

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely
correct solutions will be a pass; a few complete solutions will count more than many
partial solutions. Always carefully justify your answers. If you skip a step or omit
some details in a proof, point out the gap and, if possible, indicate what would be
required to fill it in.

Please start each solution on a new page and submit your solutions in order.

Notation
Z the commutative ring of integers.
Q the field of rational numbers.
\( \mathbb{F}_p = \mathbb{Z}/(p) \) the field of \( p \) elements where \( p \) is a prime.

1. Recall that two \( n \times n \) matrices \( A \) and \( B \) over a field are similar if there is an
invertible \( n \times n \) matrix \( Q \) so that
\[
A = QBQ^{-1}.
\]
A partition of a positive integer \( n \) is a sequence of positive integers \( n_1 \geq n_2 \geq \cdots \geq n_k \) so that
\[
n_1 + n_2 + \cdots + n_k = n.
\]
Let \( P(n) \) be the number of distinct partitions of \( n \). For example, \( P(4) = 5 \). Prove
that up to similarity of matrices, there are exactly \( P(n) \) \( n \times n \) matrices \( A \) so that
\( A^n = 0 \) (same \( n \)).

2. Describe the Galois group and the intermediate fields of the cyclotomic ex-
tension \( \mathbb{Q}(\zeta_{12})/\mathbb{Q} \).

3. 
(a) Prove that \( \mathbb{Z}[x]/(2x - 1, x^7 - 1) \cong \mathbb{Z}/(127) \).
(b) Prove that \( \mathbb{Z}[\iota]/(2\iota - 3) \cong \mathbb{F}_{13} \).

4. Let \( G \) be the group of matrices of the form
\[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
\]
with entries in the field \( \mathbb{F}_p \).
(a) Prove that \( G \) is nonabelian.
(b) Suppose \( p \) is odd. Prove that \( g^p = I_3 \) for all \( g \in G \).
(c) Suppose that \( p = 2 \). It is known that there are exactly two nonabelian
groups of order 8, up to isomorphism: the dihedral group \( D_4 \) and the
quaternion group. Assuming this fact without proof, determine which of
these groups \( G \) is isomorphic to.

5. Let \( \mathbb{F} \) be a field and let \( \mathbb{F}^\times \) denote the group of nonzero elements of \( \mathbb{F} \). Show
that every finite subgroup of \( \mathbb{F}^\times \) is cyclic.
6. (a) Let $R$ be a commutative ring with identity which is Noetherian. Let $M$ be a finitely generated $R$-module, and let $f : M \to M$ be an onto $R$-homomorphism. Prove that $f$ is an isomorphism.

(b) Give an example showing that without the assumption that $M$ is finitely generated, an onto $R$-homomorphism $f : M \to M$ is not necessarily an isomorphism.

7. Let $G$ be a finite group, $F$ an algebraically closed field, and $V$ an irreducible $F$-linear representation of $G$.

(a) Show that $\text{Hom}_F G(V, V)$ is a division algebra with $F$ in its center.

(b) Show that $V$ is finite-dimensional over $F$, and conclude that $\text{Hom}_F G(V, V)$ is also finite-dimensional.

(c) Show the inclusion $F \to \text{Hom}_F G(V, V)$ found in (a) is an isomorphism. (For $f \in \text{Hom}_F G(V, V)$, view $f$ as a linear transformation and consider $f - \alpha I$, where $\alpha$ is an eigenvalue of $f$.)

8. Suppose that $R$ is a ring. Consider an exact sequence of $R$-modules:

\[ (*) \hspace{1cm} 0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0 \]

(Recall that “exactness” means that $f$ is injective, $g$ is surjective, and $\ker(g) = \text{im}(f)$.) We say that the exact sequence $(*)$ splits if there is an $R$-module homomorphism $h : M_3 \to M_2$ such that $g \circ h : M_3 \to M_3$ is the identity map.

(a) Prove that if the exact sequence $(*)$ splits, then $M_2$ is isomorphic to the direct sum $M_1 \oplus M_3$.

(b) Suppose $R$ is the polynomial ring $C[x, y]$ in two indeterminates $x$ and $y$. Give an example of an exact sequence $(*)$ of $R$-modules which does not split.

(c) Suppose that $R$ is a commutative integral domain and that every exact sequence $(*)$ of $R$-modules splits. Prove that $R$ is a field.