2019 Algebra Prelim

September 9, 2019

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Please start each solution on a new page and submit your solutions in order.

Notation
$\mathbb{Z}$ the commutative ring of integers.
$\mathbb{Q}$ the field of rational numbers.
S_n the symmetric group on n objects.

1. Let $p, q \in \mathbb{Z}$ be two prime numbers. Prove that any group of order p^2q is solvable.

2. Let A and B be $n \times n$ matrices with entries in a field. Suppose that $A^2 = A$ and $B^2 = B$. Prove that there is an invertible matrix Q such that $Q^{-1}AQB = BQ^{-1}AQ$.

3. Find the number of intermediate fields of the cyclotomic extension $\mathbb{Q}(\zeta_{2019})/\mathbb{Q}$.

4. Let A be a commutative domain with a unity.
   (a) Prove that any flat A-module is torsion-free.
   (b) Prove that if A is a PID, then any torsion-free module is flat.
   (c) Give an example that if A is not necessarily a PID, then there may exist torsion-free, but not flat A-modules.

5. Recall the notions of *inner* and *outer automorphisms* of a group. Let G be a group and let $\mathrm{Aut}(G)$ denote the automorphism group of G. An automorphism ϕ of G is called an *inner automorphism* if there is an element $x \in G$ such that

$$\phi(g) = x^{-1}gx$$

for all $g \in G$. All inner automorphisms of G form a normal subgroup of $\mathrm{Aut}(G)$, called the *inner automorphism group* which is denoted by $\mathrm{Inn}(G)$. The quotient group $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is called the *outer automorphism group* of G, denoted by $\mathrm{Out}(G)$.

Next let G be the quaternion group, that is, the set of 8 elements $\{\pm 1, \pm i, \pm j, \pm k\}$ with group law determined by

$$(-1)i = -i = i(-1), \quad \text{and} \quad i^2 = j^2 = k^2 = -1, \quad \text{and} \quad ij = k = -ji.$$

   (a) Determine the inner automorphism group $\mathrm{Inn}(G)$.
   (b) Determine the outer automorphism group $\mathrm{Out}(G)$.

6. Let $f(x) \in \mathbb{Z}[x]$ be an irreducible polynomial of degree 5 with exactly 3 real roots and let $\mathbb{K}$ be the splitting field of $f(x)$. Prove that the Galois group $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ is isomorphic to S_5.

7. This question concerns the the group ring $\mathbb{Z}[G]$ associated to a group G. By definition, $\mathbb{Z}[G]$ is the free abelian group which has the set G as a basis. That is, $\mathbb{Z}[G]$ consists of all formal finite $\sum_{g\in G} a_g g$, where the coefficients a_g are in $\mathbb{Z}$. One makes $\mathbb{Z}[G]$ into a ring by extending the multiplication operation on G to $\mathbb{Z}[G]$ by the distributive law. As an example, if G is an infinite cyclic group generated by x, then $\mathbb{Z}[G]$ is isomorphic to the ring $\mathbb{Z}[x, x^{-1}]$, where x is regarded as an indeterminate. One basic property is the following: Suppose that G and H are groups. If one has a group homomorphism $f : G \to H$, then one obtains a ring homomorphism $F : \mathbb{Z}[G] \to \mathbb{Z}[H]$ defined by

$$F\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g f(g).$$

You do not need to prove this basic property.

Assume that G is an abelian group. Prove that $\mathbb{Z}[G]$ is a Noetherian ring if and only if G is a finitely generated group.

8. Let R be the commutative polynomial ring $\mathbb{Q}[x]$ and let S be the simple R-module $R/(x)$. Calculate $\mathrm{Hom}_R(S,S)$ and $\mathrm{Ext}^i_R(S,S)$ for each $i = 1, 2, 3, \ldots$.