2018 Algebra Prelim

September 10, 2018

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it. Please start each solution on a new page and submit your solutions in order.

1. Show that no finite group is the union of conjugates of a proper subgroup.

2. Classify all groups of order 18 up to isomorphism.

3. Let α, β denote the unique positive real 5th root of 7 and 4th root of 5, respectively. Determine the degree of $\mathbb{Q}[\alpha, \beta]$ over $\mathbb{Q}$.

4. Show that the field extension $\mathbb{Q} \subset \mathbb{Q}[\sqrt{2+\sqrt{2}}]$ is Galois and determine its Galois group.

5. Let M be a square matrix over a field K. Use a suitable canonical form to show that M is similar to its transpose M^T.

6. Let R be a commutative ring and M be an R-module.

(a) Show that $M = 0$ if and only if the localization $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of R.

(b) Find an example of a local ring R with maximal ideal $\mathfrak{m}$ and a nonzero R-module M such that $M/\mathfrak{m}M = 0$.

7. Let G be a finite group and π, π' be two irreducible representations of G. Prove or disprove the following assertion: π and π' are equivalent if and only if $\det \pi(g) = \det \pi'(g)$ for all $g \in G$.

8. Let K be a field and $R = K[x]/(x^2)$. For each integer $i \geq 0$, compute

(a) $\mathrm{Ext}^i_R(R, R/(x))$

(b) $\mathrm{Ext}^i_R(R/(x), R)$

(c) $\mathrm{Ext}^i_R(R/(x), R/(x))$.