INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it. Please start each solution on a new page and submit your solutions in order.

1. Show that no finite group is the union of conjugates of a proper subgroup.

2. Classify all groups of order 18 up to isomorphism.

3. Let $\alpha, \beta$ denote the unique positive real 5th root of 7 and 4th root of 5, respectively. Determine the degree of $\mathbb{Q}[\alpha, \beta]$ over $\mathbb{Q}$.

4. Show that the field extension $\mathbb{Q} \subset \mathbb{Q}[\sqrt{2} + \sqrt{2}]$ is Galois and determine its Galois group.

5. Let $M$ be a square matrix over a field $K$. Use a suitable canonical form to show that $M$ is similar to its transpose $M^T$.

6. Let $R$ be a commutative ring and $M$ be an $R$-module.
   
   (a) Show that $M = 0$ if and only if the localization $M_m = 0$ for all maximal ideals $m$ of $R$.

   (b) Find an example of a local ring $R$ with maximal ideal $m$ and a nonzero $R$-module $M$ such that $M/mM = 0$.

7. Let $G$ be a finite group and $\pi, \pi'$ be two irreducible representations of $G$. Prove or disprove the following assertion: $\pi$ and $\pi'$ are equivalent if and only if $\det \pi(g) = \det \pi'(g)$ for all $g \in G$.

8. Let $K$ be a field and $R = K[x]/(x^2)$. For each integer $i \geq 0$, compute
   
   (a) $\operatorname{Ext}_R^i(R, R/(x))$

   (b) $\operatorname{Ext}_R^i(R/(x), R)$

   (c) $\operatorname{Ext}_R^i(R/(x), R/(x))$. 