# 2018-19 Algebra Prelim

Spring, 2019

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it. Please start each solution on a new page and submit your solutions in order.

1. Show that the additive group $\mathbb{Q}^+$ of the rational numbers under addition has no maximal proper subgroup. Is the same true for the multiplicative group $\mathbb{Q}^*$ of nonzero rational numbers?

2. Let $p, q$ be distinct primes

(a) Show that there is at most one nonabelian group of order $pq$ up to isomorphism.

(b) Classify all pairs $(p, q)$ such that there exists a nonabelian group of order $pq$.

3. Let $\mathbb{Z}_p$ denote the cyclic group of prime order $p$.

(a) Show that $\mathbb{Z}_p$ has two irreducible representations over $\mathbb{Q}$ up to equivalence, one of dimension 1 and the other of dimension $p - 1$.

(b) Let $G$ be a finite group and $\rho : G \to GL_n(\mathbb{Q})$ be an irreducible representation of $G$ over $\mathbb{Q}$. Let $\rho_{\mathbb{C}}$ denote $\rho$ followed by the inclusion of $GL_n(\mathbb{Q})$ into $GL_n(\mathbb{C})$. We say that $\rho$ is *absolutely irreducible* if $\rho_{\mathbb{C}}$ remains irreducible over $\mathbb{C}$. Suppose that $G$ is abelian and every irreducible representation of $G$ over $\mathbb{Q}$ is absolutely irreducible. Show that $G$ is the direct product of $k$ cyclic subgroups of order 2 for some $k$.

4. Compute the splitting field and the Galois group of the polynomial $f(x) = x^5 - 3$ over the following fields: $\mathbb{Q}[e^{2\pi i/5}], \mathbb{R}$, and $\mathbb{C}$.

5. Work out the degrees of the intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}[\zeta_{12}]$, where $\zeta_{12}$ is a primitive 12th root of 1.

6. Let $R = \mathbb{Z}[x]/(x^2 + x + 1)$.

(a) Show that $R$ is Noetherian but not Artinian as a ring.

(b) Show that $R$ is an integrally closed domain.

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7. Let $R$ be a (commutative) principal ideal domain, $M, N$ finitely generated free $R$-modules, and $\phi : M \to N$ an $R$-module homomorphism.

(a) Show that the kernel $K$ of $\phi$ is a direct summand of $M$.

(b) Show by an example that the image of $\phi$ need not be a direct summand of $N$.

8. Let $R = K[x, y]$, where $K$ is a field, and let $\mathfrak{m} = (x, y) \subset R$.

(a) Find a projective resolution of the $R$-module $R/\mathfrak{m}$.

(b) Compute $\mathrm{Tor}_i^R(\mathfrak{m}, R/\mathfrak{m})$ for all $i \geq 0$ and conclude that $\mathfrak{m}$ is not a flat $R$-module.