Analysis Preliminary Exam, September 2020

Do as many of the nine problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. You may use any standard theorem from your analysis course, identifying it either by name or by stating it in full.

1. Find an explicit conformal map from the half strip $W = \{z = x + iy \,:\, x > 0,\ 0 < y < 1\}$ onto the unit disc $U = \{z \,:\, |z| < 1\}$. Your map may be given as a composition of explicit conformal maps, but be sure for each such map to give an explicit formula and state explicitly the regions being mapped.

2. Let $\{a_n\}$ be a sequence of complex numbers for which the series $\sum_{n=1}^{\infty} a_n$ converges, but does not necessarily converge absolutely. Let $f(z)$ be holomorphic in the unit disc $U = \{z \,:\, |z| < 1\}$. Prove that the series $\sum_{n=1}^{\infty} a_n f(z^n)$ converges uniformly on compact subsets of U to a holomorphic function on U.

3. Let K be a compact subset of a domain $\Omega \subset \mathbb{C}$. Let $\mathcal{F}$ be the family of all bounded holomorphic functions on Ω that have at least one zero in K. Prove that there exists a constant $C < 1$ such that, for all $f \in \mathcal{F}$, $\|f\|_K \le C\|f\|_\Omega$, where $\|f\|_K = \max_{z\in K} |f(z)|$ and $\|f\|_\Omega = \sup_{z\in\Omega} |f(z)|$.
(Hint: Consider first the subfamily $\mathcal{F}_1 = \{f \in \mathcal{F} \,:\, \|f\|_\Omega = 1\}$.)

4. A real-valued function f is said to be convex on $\mathbb{R}$ if $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$ for all $x, y \in \mathbb{R}$ and $0 \le \lambda \le 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function. For $x \in \mathbb{R}$, define the subdifferential of f at x to be the set

$$\partial f(x) = \big\{a \in \mathbb{R} \,:\, a \cdot (y - x) + f(x) \le f(y) \text{ for all } y \in \mathbb{R}\big\}.$$

For Borel sets $E \subset \mathbb{R}$, define $\partial f(E) = \bigcup_{x\in E} \partial f(x)$. Show that the set function $\mu_f(E) = m(\partial f(E))$ is a Borel measure on $\mathbb{R}$, where m is Lebesgue measure on $\mathbb{R}$.

5. Let $A \subset \mathbb{R}$ be a Lebesgue measurable set with positive and finite Lebesgue measure. Show that the support of the function $f(x) = (\chi_A * \chi_A)(x) = \int_{\mathbb{R}} \chi_A(x - y)\chi_A(y)\, dy$ contains a nonempty open set, where χ_A is the indicator function of the set A, that is, $\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space, and suppose $1 \le p < q < \infty$. Prove that $L^q(X, \mathcal{M}, \mu) \not\subset L^p(X, \mathcal{M}, \mu)$ if and only if X contains sets of arbitrarily large finite measure.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $\{f_n\}_{n=1}^{\infty}$ and f be measurable real-valued functions on X. Suppose that $f_n \to f$ a.e. and $\lim_{n\to\infty} \int |f_n|^{1/2}\, d\mu = \int |f|^{1/2}\, d\mu < \infty$. Show that $\lim_{n\to\infty} \int |f_n - f|^{1/2}\, d\mu = 0$.

8. Let $(X, \|\cdot\|_X)$ be a real Banach space. For every element $z \in X$, define

$$U_z = \big\{f \in X^* \,:\, \|f\|_{X^*} = \|z\|_X \text{ and } f(z) = \|z\|_X^2\big\},$$

where $(X^*, \|\cdot\|_{X^*})$ is the dual space of $(X, \|\cdot\|_X)$, equipped with the operator norm $\|\cdot\|_{X^*}$. Show that for each $z \in X$, the set $U_z \subset X^*$ is nonempty, convex, and closed in the operator norm topology on X^*.

9. (a) Let $\{x_n\}$ be a sequence in a normed linear space $(X, \|\cdot\|_X)$ and suppose that $x_n \to 0$ weakly. Show that the sequence $\{x_n\}$ is norm bounded, that is, there exists a constant M such that for all n, $\|x_n\|_X \le M$.

(b) Give an example of a normed linear space $(X, \|\cdot\|_X)$ and a sequence $\{f_n\}$ in the dual space $(X^*, \|\cdot\|_{X^*})$ such that $f_n \to 0$ in the weak* topology on X^* and $\|f_n\|_{X^*} \to \infty$ (with the usual operator norm on X^*).