Complex Prelim

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Please start each solution on a new page and submit your solutions in order.

1. Assume $b > 1$ is a real number. Using contour integrals compute

$$\int_0^\infty \frac{dx}{1+x^b}$$

   Express your answer as a positive real number. Justify all estimates.

2. Suppose f is a non-constant entire function and $f(1-z) = 1 - f(z)$ for all z. Prove that $f(z)$ assumes every complex number.

3. Let f and g be analytic on a connected relatively compact open set W in $\mathbb{C}$ and continuous on the closure $\overline{W}$. Prove that the maximum of $|f(z)| + |g(z)|$ occurs on the boundary ∂W.

4. (a) Find a bounded harmonic function u that is continuous on $S = \{z : |z| \leq 1, Im(z) \geq 0, z \neq 1, z \neq -1\}$ such that $u = 3$ on the interval $(-1, 1)$ of the real axis and $u = 1$ on $\{z : |z| = 1, Im(z) > 0\}$.

   (b) There are infinitely many *unbounded* harmonic functions v with the properties stated in part (a). Find two of them.

5. Is there an analytic function f that maps $\{|z| < 1\}$ into $\{|w| < 1\}$ such that $f(\frac{1}{2}) = \frac{2}{3}, f(\frac{1}{4}) = \frac{1}{3}$?

6. Prove that there is no $1-1$ holomorphic map from $D^* = \{z : 0 < |z| < 1\}$ onto $A = \{z : a < |z| < b\}$ where $0 < a < b$.

7. Suppose f and g are non-zero entire functions. Suppose also that $|f(z)| \leq |g(z)|$ when $|z| \geq 2017$. Prove that f/g is a rational function.

8. Let $\Omega = \{z : |z| < 1\} \cup \{z : |z-1| < 1\}$. Prove that there is a function f that is analytic on Ω but does not extend to *any* open set that contains Ω as a proper subset.