Complex Prelim

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

Please start each solution on a new page and submit your solutions in order.

1. Assume \( b > 1 \) is a real number. Using contour integrals compute

\[
\int_0^\infty \frac{dx}{1 + x^b}
\]

Express your answer as a positive real number. Justify all estimates.

2. Suppose \( f \) is a non-constant entire function and \( f(1-z) = 1 - f(z) \) for all \( z \). Prove that \( f(z) \) assumes every complex number.

3. Let \( f \) and \( g \) be analytic on a connected relatively compact open set \( W \) in \( \mathbb{C} \) and continuous on the closure \( \overline{W} \). Prove that the maximum of \( |f(z)| + |g(z)| \) occurs on the boundary \( \partial W \).

4. (a) Find a bounded harmonic function \( u \) that is continuous on

\[
S = \{ z : |z| \leq 1, Im(z) \geq 0, z \neq 1, z \neq -1 \}
\]

such that \( u = 3 \) on the interval \(( -1, 1) \) of the real axis and \( u = 1 \) on \( \{ z : |z| = 1, Im(z) > 0 \} \).

(b) There are infinitely many unbounded harmonic functions \( v \) with the properties stated in part (a). Find two of them.

5. Is there an analytic function \( f \) that maps \( \{|z| < 1\} \) into \( \{|w| < 1\} \) such that \( f(\frac{1}{2}) = \frac{2}{3}, f(\frac{1}{4}) = \frac{1}{3} \)?

6. Prove that there is no \( 1-1 \) holomorphic map from \( D^* = \{ z : 0 < |z| < 1 \} \) onto \( A = \{ z : a < |z| < b \} \) where \( 0 < a < b \).

7. Suppose \( f \) and \( g \) are non-zero entire functions. Suppose also that \( |f(z)| \leq |g(z)| \) when \( |z| \geq 2017 \). Prove that \( f/g \) is a rational function.

8. Let \( \Omega = \{ z : |z| < 1 \} \cup \{ z : |z - 1| < 1 \} \). Prove that there is a function \( f \) that is analytic on \( \Omega \) but does not extend to any open set that contains \( \Omega \) as a proper subset.