Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The word “smooth” means $C^\infty$, and all manifolds are assumed to be without boundary unless otherwise specified.

Please start each solution on a new page and submit your solutions in order.

1. For $n \geq 2$, let $E_n$ denote the tangent bundle of $S^n$, and let $E_n^*$ denote its one-point compactification: That is, $E_n^* = E_n \cup \{\infty\}$, with the topology whose open sets are the open subsets of $E_n$ and the sets $E_n^* \setminus K$ for compact subsets $K \subseteq E_n$. Compute the fundamental group of $E_n^*$ for each $n$.

2. Does there exist a 4-sheeted covering map from the 2-torus to the Klein bottle? Prove your answer correct.

3. Let $\mathbb{C}^*$ denote the one-point compactification of $\mathbb{C}$: $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, with the topology whose open sets are the open subsets of $\mathbb{C}$ and the sets $\mathbb{C}^* \setminus K$ for compact subsets $K \subseteq \mathbb{C}$. Give $\mathbb{C}^*$ the smooth structure determined by the atlas $\{(U, \phi), (V, \psi)\}$, where $U = \mathbb{C}, V = \mathbb{C}^* \setminus \{0\}$, and

\[
\phi(z) = z, \quad \psi(w) = \begin{cases}
1/w, & w \neq \infty, \\
0, & w = \infty.
\end{cases}
\]

(You may accept without proof that this is a smooth atlas.) Given $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, define a map $F: \mathbb{C}^* \rightarrow \mathbb{C}^*$ by

\[
F(z) = \begin{cases}
az + b, & z \neq \infty, -d/c, \\
cz + d, & z = -d/c, \\
\infty, & z = \infty, \\
a/c, & z = \infty.
\end{cases}
\]

Prove that $F$ is a diffeomorphism.

4. Let $M$ and $N$ be connected smooth manifolds, and let $\pi: M \rightarrow N$ be a smooth normal covering map. Assume that the covering automorphism group $G$ is finite.

(a) Suppose that for some fixed $k$, the manifold $M$ has the property that every smooth closed $k$-form is exact. Show that $N$ has the same property.

(b) Give a counterexample to part (a) if $G$ is infinite.
5. Let $S^3 = \{(z,w) : |z|^2 + |w|^2 = 1\}$ denote the unit sphere in $\mathbb{C}^2$, and define a flow on $S^3$ by

$$\theta_t(z,w) = (e^{it}z, e^{it}w).$$

Let $X$ denote the infinitesimal generator of this flow. Does there exist a closed 1-form $\eta$ on $S^3$ such that $\eta(X) \equiv 1$ everywhere on $S^3$? Prove your answer correct.

6. Define a multiplication on $\mathbb{R} \times \mathbb{R}^+$ by

$$(x,y) \cdot (x',y') = (x + yx', yy').$$

Show that this makes $\mathbb{R} \times \mathbb{R}^+$ into a Lie group, and find the left-invariant vector fields and 1-parameter subgroups.

7. Define vector fields $V, W$ on $\mathbb{R}^3$ by

$$V = -3z^2 \frac{\partial}{\partial x} + 2x \frac{\partial}{\partial z}, \quad W = -3z^2 \frac{\partial}{\partial y} + 2y \frac{\partial}{\partial z}.$$

(a) Is there a 2-dimensional submanifold $M \subset \mathbb{R}^3$ containing the point $(0,0,1)$ such that both $V$ and $W$ are tangent to $M$? Prove your answer correct.

(b) Is there a nonconstant smooth real-valued function $f$ defined on a connected neighborhood of $(0,0,1)$ that satisfies the following partial differential equations?

$$-3z^2 \frac{\partial f}{\partial x} + 2x \frac{\partial f}{\partial z} = -3z^2 \frac{\partial f}{\partial y} + 2y \frac{\partial f}{\partial z} = 0.$$

If no, prove there is no such function $f$. If yes, discuss what additional conditions (if any) are needed to make the solution $f$ unique, and justify your answer.

8. Let $X \subseteq M_n(\mathbb{R})$ denote the set of orthogonal idempotents of rank $k$: That is, $X$ consists of all $n \times n$ real matrices $A$ such that (i) rank($A$) = $k$, (ii) $A^2 = A$, and (iii) the image and kernel of $A$ are orthogonal subspaces. Show that $X$ is a smooth embedded submanifold diffeomorphic to the Grassmannian $G_k(\mathbb{R}^n)$. (Here, $G_k(\mathbb{R}^n)$ is the set of $k$-dimensional linear subspaces of $\mathbb{R}^n$, with the smooth manifold structure for which the natural action of $GL(n, \mathbb{R})$ is smooth).