Topology and Geometry of Manifolds Preliminary Exam
September 2017

Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in.

The word "smooth" means C^∞, and all manifolds are assumed to be without boundary unless otherwise specified.

Please start each solution on a new page and submit your solutions in order.

1. For $n \geq 2$, let E_n denote the tangent bundle of $\mathbb{S}^n$, and let E_n^* denote its one-point compactification: That is, $E_n^* = E_n \cup \{\infty\}$, with the topology whose open sets are the open subsets of E_n and the sets $E_n^* \setminus K$ for compact subsets $K \subseteq E_n$. Compute the fundamental group of E_n^* for each n.

2. Does there exist a 4-sheeted covering map from the 2-torus to the Klein bottle? Prove your answer correct.

3. Let $\mathbb{C}^*$ denote the one-point compactification of $\mathbb{C}$: $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, with the topology whose open sets are the open subsets of $\mathbb{C}$ and the sets $\mathbb{C}^* \setminus K$ for compact subsets $K \subseteq \mathbb{C}$. Give $\mathbb{C}^*$ the smooth structure determined by the atlas $\{(U, \phi),\ (V, \psi)\}$, where $U = \mathbb{C}$, $V = \mathbb{C}^* \setminus \{0\}$, and

$$\phi(z) = z, \qquad \psi(w) = \begin{cases} \dfrac{1}{w}, & w \neq \infty, \\ 0, & w = \infty. \end{cases}$$

   (You may accept without proof that this is a smooth atlas.) Given $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, define a map $F \colon \mathbb{C}^* \to \mathbb{C}^*$ by

$$F(z) = \begin{cases} \dfrac{az + b}{cz + d}, & z \neq \infty,\ -d/c, \\ \infty, & z = -d/c, \\ a/c, & z = \infty. \end{cases}$$

   Prove that F is a diffeomorphism.

4. Let M and N be connected smooth manifolds, and let $\pi \colon M \to N$ be a smooth normal covering map. Assume that the covering automorphism group G is finite.
   (a) Suppose that for some fixed k, the manifold M has the property that every smooth closed k-form is exact. Show that N has the same property.
   (b) Give a counterexample to part (a) if G is infinite.

5. Let $\mathbb{S}^3 = \{(z,w) : |z|^2 + |w|^2 = 1\}$ denote the unit sphere in $\mathbb{C}^2$, and define a flow on $\mathbb{S}^3$ by

$$\theta_t(z,w) = (e^{it}z,\ e^{it}w).$$

Let X denote the infinitesimal generator of this flow. Does there exist a closed 1-form η on $\mathbb{S}^3$ such that $\eta(X) \equiv 1$ everywhere on $\mathbb{S}^3$? Prove your answer correct.

6. Define a multiplication on $\mathbb{R} \times \mathbb{R}^+$ by

$$(x,y) \cdot (x',y') = (x + yx', yy').$$

Show that this makes $\mathbb{R} \times \mathbb{R}^+$ into a Lie group, and find the left-invariant vector fields and 1-parameter subgroups.

7. Define vector fields V, W on $\mathbb{R}^3$ by

$$V = -3z^2\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial z}, \qquad W = -3z^2\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}.$$

   (a) Is there a 2-dimensional submanifold $M \subset \mathbb{R}^3$ containing the point $(0,0,1)$ such that both V and W are tangent to M? Prove your answer correct.

   (b) Is there a nonconstant smooth real-valued function f defined on a connected neighborhood of $(0,0,1)$ that satisfies the following partial differential equations?

$$-3z^2\frac{\partial f}{\partial x} + 2x\frac{\partial f}{\partial z} = -3z^2\frac{\partial f}{\partial y} + 2y\frac{\partial f}{\partial z} = 0.$$

   If no, prove there is no such function f. If yes, discuss what additional conditions (if any) are needed to make the solution f unique, and justify your answer.

8. Let $X \subseteq M_n(\mathbb{R})$ denote the set of orthogonal idempotents of rank k: That is, X consists of all $n \times n$ real matrices A such that (i) $\operatorname{rank}(A) = k$, (ii) $A^2 = A$, and (iii) the image and kernel of A are orthogonal subspaces. Show that X is a smooth embedded submanifold diffeomorphic to the Grassmannian $G_k(\mathbb{R}^n)$. (Here, $G_k(\mathbb{R}^n)$ is the set of k-dimensional linear subspaces of $\mathbb{R}^n$, with the smooth manifold structure for which the natural action of $GL(n,\mathbb{R})$ is smooth).